

Nash equilibrium payoffs for stochastic differential games with jumps and coupled nonlinear cost functionals*

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August 19, 2011

Abstract: In this paper we investigate Nash equilibrium payoffs for two-player nonzero-sum stochastic differential games whose cost functionals are defined by a system of coupled backward stochastic differential equations. We obtain an existence theorem and a characterization theorem for Nash equilibrium payoffs. For this end the problem is described equivalently by a stochastic differential game with jumps. But, however, unlike the work by Buckdahn, Hu and Li [9], here the important tool of a dynamic programming principle for stopping times has to be developed. Moreover, we prove that the lower and upper value functions are the viscosity solutions of the associated coupled systems of PDEs of Isaacs type, respectively. Our results generalize those by Buckdahn, Cardaliaguet and Rainer [7] and by Lin [16].

Keywords: stochastic differential game; Nash equilibrium payoff; backward stochastic differential equation; dynamic programming principle; dynamic programming principle for stopping times; coupled systems of Isaacs equations.

1 Introduction

The objective of this paper is to study Nash equilibrium payoffs for two-player nonzero-sum stochastic differential games (SDGs, for short) with jumps and coupled nonlinear cost functionals. Since the pioneering work of Fleming and Souganidis [11], SDGs have been studied by many authors. For instance, recently, Buckdahn and Li [10] investigated zero-sum two-player SDGs with nonlinear cost functionals using a backward stochastic differential equation (BSDE, for short) approach. Unlike Fleming and Souganidis [11] they allow the controls to depend on the past and prove with a Girsanov transformation argument that the priori random value functions are deterministic. Buckdahn, Hu and Li [9] extended the approach developed in [10] to SDGs with jumps, while Biswas [5] investigated two-player zero-sum SDGs with jump-diffusion in the sense of Fleming and Souganidis [11]. The reader interested in other approaches can be also referred to Bayraktar and Poor [3], Hamadène [12], Karatzas and Zamfirescu [14], Tang and Hou [23], and the references therein.

In nonzero-sum SDGs, Hamadène, Lepeltier and Peng [13] obtained the existence of a Nash equilibrium point for nonzero sum SDGs with the help of BSDEs. Bessoussan and Frehse [4] and Mannucci [19] obtained Nash equilibrium payoffs for SDGs by using parabolic partial differential equations. But both methods rely heavily on the assumption of the non degeneracy diffusion of the coefficient and it is independence of controls. Buckdahn, Cardaliaguet and Rainer [7] got rid of the strong assumptions on the diffusion coefficient. In Lin [16], we generalize the result in [7]

*This work is supported by the Marie Curie Initial Training Network (PITN-GA-2008-213841).

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by investigating Nash equilibrium payoffs for nonzero-sum SDGs with nonlinear cost functionals. However, in [16] the cost functionals of the both players are defined by a system of decoupled BSDEs. An open problem was how to study SDGs whose cost functionals are defined by two coupled BSDEs. This is the objective of this paper. Our method is based on an implementation of the approach by Pardoux, Pradeilles and Rao [20] into the framework of SDGs.

Let us be more precise now: The dynamics of our two-player nonzero-sum SDG is given by the process $N^{t,i}$ and the following doubly controlled stochastic system:

$$\begin{cases} dX_s^{t,x;u,v} &= b(s, X_s^{t,x;u,v}, u_s, v_s)ds + \sigma(s, X_s^{t,x;u,v}, u_s, v_s)dB_s, \\ X_t^{t,x;u,v} &= x, \end{cases} \quad s \in [t, T],$$

where $\{B_t\}_{t \geq 0}$ is a d -dimensional standard Wiener process, $\{N_t\}_{t \geq 0}$ is a Poisson process independent of $\{B_t\}_{t \geq 0}$, and \mathbb{F} is the filtration generated by B and N . For $0 \leq s \leq t \leq T, i = 1, 2$, we let $N_s^{t,i} = m(i + N_s - N_t)$, where $m(j) = 1$, if j is odd, and $m(j) = 2$, if j is even. The control $u = \{u\}_{s \in [t, T]}$ (resp. $v = \{v\}_{s \in [t, T]}$) is supposed to \mathbb{F} -predictable and takes its values in a compact metric space U (resp. V). The set of these controls is denoted by $\mathcal{U}_{t,T}$ (resp. $\mathcal{V}_{t,T}$). We shall give its assumptions on b and σ in the next section.

We define our nonlinear cost functionals by introducing a system of two coupled BSDEs:

$$\begin{cases} -d^1 \tilde{Y}_s &= \tilde{f}_1(s, X_s^{t,x;u,v}, {}^1\tilde{Y}_s, {}^2\tilde{Y}_s + {}^2\tilde{H}_s, {}^1\tilde{Z}_s, u_s, v_s)ds - {}^1\tilde{Z}_s dB_s - {}^1\tilde{H}_s dN_s, \\ -d^2 \tilde{Y}_s &= \tilde{f}_2(s, X_s^{t,x;u,v}, {}^1\tilde{Y}_s + {}^1\tilde{H}_s, {}^2\tilde{Y}_s, {}^2\tilde{Z}_s, u_s, v_s)ds - {}^2\tilde{Z}_s dB_s - {}^2\tilde{H}_s dN_s, \\ {}^1\tilde{Y}_T &= \Phi_1(X_T^{t,x;u,v}), {}^2\tilde{Y}_T = \Phi_2(X_T^{t,x;u,v}), \end{cases} \quad s \in [t, T]. \quad (1.1)$$

The assumptions on Φ_i and $\tilde{f}_i, i = 1, 2$, will be given in the next section. The cost functional for the i^{th} player, $i = 1, 2$, is defined by

$$J_i(t, x; u, v) := {}^i\tilde{Y}_t^{t,x;u,v}, \quad (t, x) \in [0, T] \times \mathbb{R}^n,$$

where $({}^i\tilde{Y}_t^{t,x;u,v}, {}^i\tilde{Z}_t^{t,x;u,v}, {}^i\tilde{H}_t^{t,x;u,v}), i = 1, 2$, is the unique solution of (1.1). Note the special form of \tilde{f}_1 and \tilde{f}_2 , which is related with our approach. The general case of \tilde{f}_i not depending on ${}^i\tilde{H}$ is still open.

In our framework, in opposite to zero-sum SDGs, nonzero-sum SDGs are of the type of "NAD strategy against NAD strategy": an NAD strategy is a measurable, nonanticipative mapping $\alpha : \mathcal{V}_{t,T} \rightarrow \mathcal{U}_{t,T}$ for the 1^{th} player (resp. $\beta : \mathcal{U}_{t,T} \rightarrow \mathcal{V}_{t,T}$ for the 2^{th} player) and has a delay (The precise definition will be introduced in next section). The set of all such NAD strategies for 1^{th} player is denoted by $\mathcal{A}_{t,T}$ (resp. for 2^{th} player is denoted by $\mathcal{B}_{t,T}$).

For $(\alpha, \beta) \in \mathcal{A}_{t,T} \times \mathcal{B}_{t,T}$, there exists a unique couple of controls $(u, v) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$ such that $(\alpha(v), \beta(u)) = (u, v)$. This allows to define $J_i(t, x; \alpha, \beta) := J_i(t, x; u, v)$, as well as the value functions of the two-player zero-sum SDG associated with $J_i, i = 1, 2$, the lower value function

$$W_i(t, x) := \text{esssup}_{\alpha \in \mathcal{A}_{t,T}} \text{essinf}_{\beta \in \mathcal{B}_{t,T}} J_i(t, x; \alpha, \beta),$$

and the upper value function

$$U_i(t, x) := \text{essinf}_{\beta \in \mathcal{B}_{t,T}} \text{esssup}_{\alpha \in \mathcal{A}_{t,T}} J_i(t, x; \alpha, \beta), i = 1, 2.$$

We note that, since the BSDEs (1.1) are coupled, the values of the two-player zero-sum SDGs are also coupled.

Pardoux, Pradeilles and Rao [20] gave a probabilistic interpretation of a coupled system of semi-linear parabolic partial differential equations, and Buckdahn and Hu [8] obtained a probabilistic interpretation of a coupled system of Hamilton-Jacobi-Bellman equations. In our

approach we need a probabilistic interpretation of coupled systems of Hamilton-Jacobi-Bellman-Isaacs equations: A first result of this paper is that the value functions $U = (U_1, U_2)$ and $W = (W_1, W_2)$ are viscosity solutions of the following coupled Isaacs equations:

$$\begin{cases} \frac{\partial}{\partial t} U_i(t, x) + H_i^+(t, x, U_1(t, x), U_2(t, x), DU_i(t, x), D^2 U_i(t, x)) &= 0, & (t, x) \in [0, T] \times \mathbb{R}^n, \\ U_i(T, x) &= \Phi_i(x), & i = 1, 2, \end{cases}$$

and

$$\begin{cases} \frac{\partial}{\partial t} W_i(t, x) + H_i^-(t, x, W_1(t, x), W_2(t, x), DW_i(t, x), D^2 W_i(t, x)) &= 0, & (t, x) \in [0, T] \times \mathbb{R}^n, \\ W_i(T, x) &= \Phi_i(x), & i = 1, 2, \end{cases}$$

respectively, where, for $(t, x, y_1, y_2, p, A, u, v) \in [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d \times U \times V$,

$$H_i(t, x, y_1, y_2, p, A, u, v) = \frac{1}{2} \text{tr}(\sigma \sigma^T(t, x, u, v) A) + p b(t, x, u, v) + \tilde{f}_i(t, x, y_1, y_2, p \sigma(t, x, u, v), u, v),$$

and

$$\begin{aligned} H_i^-(t, x, y_1, y_2, p, A) &= \sup_{u \in U} \inf_{v \in V} H_i(t, x, y_1, y_2, p, A, u, v), \\ H_i^+(t, x, y_1, y_2, p, A) &= \inf_{v \in V} \sup_{u \in U} H_i(t, x, y_1, y_2, p, A, u, v). \end{aligned}$$

A crucial step in the proof of this results is a dynamic programming principle for stopping times: For any stopping time τ with $0 \leq t < \tau \leq T$, $x \in \mathbb{R}^n, i = 1, 2$,

$$\begin{aligned} W_i(t, x) &= \text{esssup}_{\alpha \in \mathcal{A}_{t, \tau}} \text{essinf}_{\beta \in \mathcal{B}_{t, \tau}} {}^i G_{t, \tau}^{t, x; \alpha, \beta} [W_{N_\tau^{t, i}}(\tau, X_\tau^{t, x; \alpha, \beta})], \\ U_i(t, x) &= \text{essinf}_{\beta \in \mathcal{B}_{t, \tau}} \text{esssup}_{\alpha \in \mathcal{A}_{t, \tau}} {}^i G_{t, \tau}^{t, x; \alpha, \beta} [U_{N_\tau^{t, i}}(\tau, X_\tau^{t, x; \alpha, \beta})], \end{aligned}$$

where ${}^i G_{t, \tau}^{t, x; \alpha, \beta}[\cdot]$ is a backward stochastic semigroup (the precise definition as well as those of $\mathcal{A}_{t, \tau}$ and $\mathcal{B}_{t, \tau}$ will be given later).

The most important part of this paper is dedicated to the Nash equilibrium payoffs of our games. A couple $(e_1, e_2) \in \mathbb{R}^2$ is called a Nash equilibrium payoff at the point (t, x) , if for any $\varepsilon > 0$, there exists $(\alpha_\varepsilon, \beta_\varepsilon) \in \mathcal{A}_{t, T} \times \mathcal{B}_{t, T}$ such that, for all $(\alpha, \beta) \in \mathcal{A}_{t, T} \times \mathcal{B}_{t, T}$,

$$J_1(t, x; \alpha_\varepsilon, \beta_\varepsilon) \geq J_1(t, x; \alpha, \beta_\varepsilon) - \varepsilon, \quad J_2(t, x; \alpha_\varepsilon, \beta_\varepsilon) \geq J_2(t, x; \alpha_\varepsilon, \beta) - \varepsilon, \quad \mathbb{P} - a.s.,$$

and

$$|\mathbb{E}[J_j(t, x; \alpha_\varepsilon, \beta_\varepsilon)] - e_j| \leq \varepsilon, \quad j = 1, 2.$$

The main results of this paper concerns the existence and a characterization of Nash equilibrium payoffs for our games: Under Isaacs condition (which will be precised later), a couple $(e_1, e_2) \in \mathbb{R}^2$ is a Nash equilibrium payoff at point (t, x) if and only if for all $\varepsilon > 0$, there exists $(u^\varepsilon, v^\varepsilon) \in \mathcal{U}_{t, T} \times \mathcal{V}_{t, T}$ such that for all stopping time τ independent of \mathcal{F}_t with $\tau \in [t, T]$ and $j = 1, 2$,

$$\mathbb{P}\left({}^{N_\tau^{t, j}} \tilde{Y}_\tau^{t, x; u^\varepsilon, v^\varepsilon} \geq W_{N_\tau^{t, j}}(\tau, X_\tau^{t, x; u^\varepsilon, v^\varepsilon}) - \varepsilon \mid \mathcal{F}_t \right) \geq 1 - \varepsilon, \quad \mathbb{P} - a.s.,$$

and

$$|\mathbb{E}[J_j(t, x; u^\varepsilon, v^\varepsilon)] - e_j| \leq \varepsilon.$$

This result turn out to be the key of the proof of a Nash equilibrium exists, for all $(t, x) \in [0, T] \times \mathbb{R}^n$.

Let us explain what is new and which difficulties are related with. In comparison with [7] and [16], the first difficulty was to get a dynamic programming principle for a system of two coupled BSDEs. To overcome this difficulty, we associate with this system an auxiliary one which cost functionals coincide with ours. This leads to the new problem is that we need a dynamic programming principle for this system not only for deterministic but also for stopping times. The method used in Buckdahn and Hu [8] to get for control problems the dynamic programming principle on stopping times is not applicable any more, because in the framework of SDGs the monotonicity arguments used in [8] doesn't work any more. Another technical difficulty comes from the fact that we study here nonzero-sum SDGs and not zero-sum SDGs. In order to give both players symmetric tools, they have to use "strategies against strategies" and not only "strategies against controls" as in [10]. Finally, comparing to our previous work [16], the presence of jump terms adds a supplementary complexity. And related works see [17] and [18].

This paper is organized as follows. In Section 2 we introduce some notations and recall some basics of BSDEs with jumps, which will be needed in what follows. Section 3 introduces the setting of SDGs and studies the dynamic programming principle for stopping times. Section 4 gives a probabilistic interpretation of coupled systems of Isaacs equations. In Section 5 we investigate Nash equilibrium payoffs for nonzero-sum SDGs. An existence theorem and a characterization theorem of Nash equilibrium payoffs are established. Finally, we postpone the proof of the Theorems 3.10 and 4.2 to Section 6.

2 Preliminaries

The objective of this section is to give some notations and to recall some basic results of BSDEs with jumps, which will be useful in what follows.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete probability space. The filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$ is generated by the following two stochastic processes defined on $(\Omega, \mathcal{F}, \mathbb{P})$, and augmented by all \mathbb{P} -null sets:

- $\{B_t\}_{t \geq 0}$ is a d -dimensional standard Wiener process;
- $\{N_t\}_{t \geq 0}$ is a Poisson process independent of $\{B_t\}_{t \geq 0}$ such that $\{\tilde{N}_t = N_t - \lambda t\}_{t \geq 0}$ is an $\{\mathcal{F}_t\}_{t \geq 0}$ -martingale, for some $\lambda > 0$.

Let $T > 0$ be an arbitrarily fixed time horizon. For any $n \geq 1$, we denote by $|z|$ the Euclidean norm of $z \in \mathbb{R}^n$. We introduce the following spaces of stochastic processes.

- $L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^n) = \left\{ \xi \mid \xi : \Omega \rightarrow \mathbb{R}^n \text{ is an } \mathcal{F}_T\text{-measurable random variable such that } \mathbb{E}[|\xi|^2] < +\infty \right\},$
- $S^2(0, T; \mathbb{R}) = \left\{ \varphi \mid \varphi : \Omega \times [0, T] \rightarrow \mathbb{R} \text{ is an } \mathbb{F}\text{-adapted càdlàg process such that } \mathbb{E} \left[\sup_{0 \leq t \leq T} |\varphi_t|^2 \right] < +\infty \right\},$
- $\mathcal{H}^2(0, T; \mathbb{R}^d) = \left\{ \varphi \mid \varphi : \Omega \times [0, T] \rightarrow \mathbb{R}^d \text{ is an } \mathbb{F}\text{-predictable process such that } \mathbb{E} \int_0^T |\varphi_t|^2 dt < +\infty \right\}.$

Let us consider the following BSDE with data (f, ξ) :

$$y_t = \xi + \int_t^T f(s, y_s, z_s, k_s) ds - \int_t^T z_s dB_s - \int_t^T k_s d\tilde{N}_s, \quad 0 \leq t \leq T. \quad (2.1)$$

Here $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$ is \mathbb{F} -predictable and satisfies the following assumptions:

(H1) (Lipschitz condition): There exists a positive constant C such that, for all $(t, y_i, z_i, k_i) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$, $i = 1, 2$,

$$|f(t, y_1, z_1, k_1) - f(t, y_2, z_2, k_2)| \leq C(|y_1 - y_2| + |z_1 - z_2| + |k_1 - k_2|).$$

(H2) $f(\cdot, 0, 0, 0) \in \mathcal{H}^2(0, T; \mathbb{R})$.

(H3) There exists a constant $K > -1$ such that, for all $(t, y, z, k_1, k_2) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^2$,

$$f(t, y, z, k_1) - f(t, y, z, k_2) \leq K(k_1 - k_2).$$

We have the following existence and uniqueness theorem of BSDE (2.1). For its proof we refer the reader to Tang and Li [24].

Lemma 2.1. *Let the assumptions (H1) and (H2) hold. Then, for all $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$, BSDE (2.1) has a unique solution $(y, z, k) \in S^2(0, T; \mathbb{R}) \times \mathcal{H}^2(0, T; \mathbb{R}^d) \times \mathcal{H}^2(0, T; \mathbb{R})$.*

We have the following comparison theorem for solutions of BSDEs (2.1), which is proved with the help of standard arguments (see Royer [22]).

Lemma 2.2. *Let us denote by (y^1, z^1, k^1) and (y^2, z^2, k^2) the solutions of BSDEs with data (f^1, ξ^1) and (f^2, ξ^2) , respectively. Moreover, if $\xi^1, \xi^2 \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$, and f^1 and f^2 satisfy the assumptions (H1), (H2) and (H3), and the following holds*

$$(i) \xi^1 \leq \xi^2, \mathbb{P} - a.s.,$$

$$(ii) f^1(t, y_t^2, z_t^2, k_t^2) \leq f^2(t, y_t^2, z_t^2, k_t^2), \quad dtd\mathbb{P} - a.e.,$$

then we have $y_t^1 \leq y_t^2$, a.s., for all $t \in [0, T]$.

For some $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$ satisfying (H1) and (H2), we let, for $i = 1, 2$,

$$f_i(s, y_s^i, z_s^i, k_s^i) = f(s, y_s^i, z_s^i, k_s^i) + \varphi_i(s),$$

where $\varphi_i \in \mathcal{H}^2(0, T; \mathbb{R})$. If ξ_1 and ξ_2 are in $L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$, then we have the following lemma.

Lemma 2.3. *Let us denote by (y^1, z^1, k^1) and (y^2, z^2, k^2) the solutions of BSDE (2.1) with the data (ξ_1, f_1) and (ξ_2, f_2) , respectively. Then the following holds: for all $t \in [0, T]$,*

$$\begin{aligned} |y_t^1 - y_t^2|^2 &+ \frac{1}{2} \mathbb{E} \left[\int_t^T e^{\beta(s-t)} (|y_s^1 - y_s^2|^2 + |z_s^1 - z_s^2|^2) ds \middle| \mathcal{F}_t \right] + \frac{\lambda}{2} \mathbb{E} \left[\int_t^T e^{\beta(s-t)} |k_s^1 - k_s^2|^2 ds \middle| \mathcal{F}_t \right] \\ &\leq \mathbb{E} [e^{\beta(T-t)} |\xi_1 - \xi_2|^2 \middle| \mathcal{F}_t] + \mathbb{E} \left[\int_t^T e^{\beta(s-t)} |\varphi_1(s) - \varphi_2(s)|^2 ds \middle| \mathcal{F}_t \right], \quad \mathbb{P} - a.s. \end{aligned}$$

Here $\beta \geq 2 + 2C + 4C^2$, where C is the Lipschitz constant in (H1).

For the proof, the readers can be referred to Barles, Buckdahn and Pardoux [1].

3 Stochastic differential games with jumps

In this section, we first introduce nonzero-sum SDGs, and then we define the value functions and show that they have a deterministic version. Finally, we state the dynamic programming principle for stopping times, which is crucial for the next section.

Let \mathcal{U} (resp., \mathcal{V}) be the set of admissible control processes for the first (resp., second) player, i.e., the set of all U (resp., V)-valued \mathbb{F} -predictable processes. We suppose that the control state spaces U and V are compact metric spaces.

For given admissible controls $u(\cdot) \in \mathcal{U}$ and $v(\cdot) \in \mathcal{V}$, we consider the following stochastic differential equation (SDE): for $t \in [0, T]$ and $\zeta \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^n)$,

$$\begin{cases} dX_s^{t, \zeta; u, v} &= b(s, X_s^{t, \zeta; u, v}, u_s, v_s)ds + \sigma(s, X_s^{t, \zeta; u, v}, u_s, v_s)dB_s, & s \in [t, T], \\ X_t^{t, \zeta; u, v} &= \zeta, \end{cases} \quad (3.1)$$

where

$$b : [0, T] \times \mathbb{R}^n \times U \times V \rightarrow \mathbb{R}^n, \quad \sigma : [0, T] \times \mathbb{R}^n \times U \times V \rightarrow \mathbb{R}^{n \times d},$$

satisfy the following assumptions:

- (i) For every fixed $x \in \mathbb{R}^n$, $b(\cdot, x, \cdot, \cdot)$, and $\sigma(\cdot, x, \cdot, \cdot)$ are continuous in (t, u, v) .
- (ii) There exists a constant $C > 0$ such that, for all $t \in [0, T]$, $x, x' \in \mathbb{R}^n$, $u \in U$, $v \in V$,
$$|b(t, x, u, v) - b(t, x', u, v)| + |\sigma(t, x, u, v) - \sigma(t, x', u, v)| \leq C|x - x'|. \quad (H4)$$

From (H4) we know that there exists some $C > 0$ such that, for all $0 \leq t \leq T$, $u \in U$, $v \in V$, $x \in \mathbb{R}^n$,

$$|b(t, x, u, v)| + |\sigma(t, x, u, v)| \leq C(1 + |x|).$$

It is well known that under (H4), for any $u(\cdot) \in \mathcal{U}$ and $v(\cdot) \in \mathcal{V}$, SDE (3.1) has a unique strong solution. Furthermore, we have the following estimates for the solution of SDE (3.1) (e.g., see [9]).

Proposition 3.1. *Let the assumption (H4) hold. Then, for $p \geq 2$, there exists a positive constant $C = C_p$ such that, for $t \in [0, T]$, $u(\cdot) \in \mathcal{U}$, $v(\cdot) \in \mathcal{V}$ and $\zeta, \zeta' \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^n)$,*

$$\begin{aligned} \mathbb{E} \left[\sup_{s \in [t, T]} |X_s^{t, \zeta; u, v} - X_s^{t, \zeta'; u, v}|^p \middle| \mathcal{F}_t \right] &\leq C|\zeta - \zeta'|^p, \quad \mathbb{P} - a.s., \\ \mathbb{E} \left[\sup_{s \in [t, T]} |X_s^{t, \zeta; u, v} - \zeta|^p \middle| \mathcal{F}_t \right] &\leq C(1 + |\zeta|^p)|T - t|^{\frac{p}{2}}, \quad \mathbb{P} - a.s. \end{aligned} \quad (3.2)$$

For given $\Phi_i : \mathbb{R}^n \rightarrow \mathbb{R}$, $\tilde{f}_i : [0, T] \times \mathbb{R}^n \times \mathbb{R}^2 \times \mathbb{R}^d \times U \times V \rightarrow \mathbb{R}$, $i = 1, 2$, we make the following assumptions:

- (i) For every fixed $(x, y, z) \in \mathbb{R}^n \times \mathbb{R}^2 \times \mathbb{R}^d \times \mathbb{R}$, $\tilde{f}_i(\cdot, x, y, z, \cdot, \cdot)$ is continuous in (t, u, v) and there exists a constant $C > 0$ such that, for all $t \in [0, T]$, $x, x' \in \mathbb{R}^n$, $y, y' \in \mathbb{R}^2$, $z, z' \in \mathbb{R}^d$, $u \in U$ and $v \in V$,
$$|\tilde{f}_i(t, x, y, z, u, v) - \tilde{f}_i(t, x', y', z', u, v)| \leq C(|x - x'| + |y - y'| + |z - z'|).$$
- (ii) For all $(y_1, y_2), (y'_1, y'_2) \in \mathbb{R}^2$, and $(t, x, z, u, v) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^d \times U \times V$, there exists a constant $K > -1$ such that
$$\begin{aligned} \tilde{f}_1(t, x, y_1, y_2, z, u, v) - \tilde{f}_1(t, x, y'_1, y'_2, z, u, v) &\geq K(y_2 - y'_2), \\ \tilde{f}_2(t, x, y_1, y_2, z, u, v) - \tilde{f}_2(t, x, y'_1, y'_2, z, u, v) &\geq K(y_1 - y'_1). \end{aligned}$$
- (iii) There exists a constant $C > 0$ such that, for all $x, x' \in \mathbb{R}^n$, $i = 1, 2$,
$$|\Phi_i(x) - \Phi_i(x')| \leq C|x - x'|. \quad (H5)$$

The following system of two coupled BSDEs will define the cost functionals of the game associated with (3.1).

$$\begin{cases} -d \, {}^1\tilde{Y}_s &= \tilde{f}_1(s, X_s^{t,x;u,v}, {}^1\tilde{Y}_s, {}^2\tilde{Y}_s + {}^2\tilde{H}_s, {}^1\tilde{Z}_s, u_s, v_s)ds - \lambda \, {}^1\tilde{H}_s ds - {}^1\tilde{Z}_s dB_s - {}^1\tilde{H}_s d\tilde{N}_s, \\ -d \, {}^2\tilde{Y}_s &= \tilde{f}_2(s, X_s^{t,x;u,v}, {}^1\tilde{Y}_s + {}^1\tilde{H}_s, {}^2\tilde{Y}_s, {}^2\tilde{Z}_s, u_s, v_s)ds - \lambda \, {}^2\tilde{H}_s ds - {}^2\tilde{Z}_s dB_s - {}^2\tilde{H}_s d\tilde{N}_s, \\ {}^1\tilde{Y}_T &= \Phi_1(X_T^{t,x;u,v}), \, {}^2\tilde{Y}_T = \Phi_2(X_T^{t,x;u,v}), \, s \in [t, T], \end{cases} \quad (3.3)$$

where $X^{t,x;u,v}$ is the solution of equation (3.1) with $\zeta = x \in \mathbb{R}^n$. Under the assumption (H5), from Tang and Li [24] we know that equation (3.3) has a unique solution. For given control processes $u(\cdot) \in \mathcal{U}$ and $v(\cdot) \in \mathcal{V}$, we introduce now the associated cost functional for the i^{th} player, $i = 1, 2$,

$$J_i(t, x; u, v) := {}^i\tilde{Y}_t^{t,x;u,v}, \quad (t, x) \in [0, T] \times \mathbb{R}^n,$$

where $(\tilde{Y}^{t,x;u,v}, \tilde{Z}^{t,x;u,v}, \tilde{H}^{t,x;u,v})$ is the solution of (3.3).

In the above definition we generalize the framework studied by Lin[16]. Indeed, in [16] we studied cost functionals defined by a decoupled system of BSDEs, while now the both BSDEs are coupled: the both players do not only influence mutually their cost functionals in the choice of their control processes, but also their gain processes ${}^i\tilde{Y}^{t,x;u,v}$, $i = 1, 2$. With an argument introduced in Pardoux, Pradeilles and Rao [20] we can transfer the coupled system of BSDEs to a decoupled system of BSDEs. For $0 \leq s \leq t \leq T$, $i = 1, 2$, we denote by $N((0, s]) := N_s$ and $N((t, s]) := N_s - N_t$. Let us define a Markov process $N_s^{t,i}$ as follows: $N_s^{t,i} = m(i + N((t, s]))$, where $m(j) = 1$, if j is odd, and $m(j) = 2$, if j is even.

For $(t, x, y, h, z, u, v) \in [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times U \times V$, we define

$$\begin{aligned} f_1(t, x, y, h, z, u, v) &:= \tilde{f}_1(t, x, y, y + h, z, u, v), \\ f_2(t, x, y, h, z, u, v) &:= \tilde{f}_2(t, x, y + h, y, z, u, v), \end{aligned}$$

and we consider the following controlled decoupled BSDEs with jumps: for $i = 1, 2$,

$$\begin{cases} -d \, {}^iY_s^{t,x;u,v} &= f_{N_s^{t,i}}(s, X_s^{t,x;u,v}, {}^iY_s^{t,x;u,v}, {}^iH_s^{t,x;u,v}, {}^iZ_s^{t,x;u,v}, u_s, v_s)ds, \\ &\quad -\lambda \, {}^iH_s^{t,x;u,v} ds - {}^iZ_s^{t,x;u,v} dB_s - {}^iH_s^{t,x;u,v} d\tilde{N}_s, \, s \in [t, T], \\ {}^iY_T^{t,x;u,v} &= \Phi_{N_T^{t,i}}(X_T^{t,x;u,v}). \end{cases} \quad (3.4)$$

Since \tilde{f}_i , $i = 1, 2$, are Lipschitz in (x, y, z) , uniformly with respect to (t, u, v) , it is easy to check that also the coefficients f_i , $i = 1, 2$, have this property. From Lemma 2.1 we know that the above BSDE has a unique solution. In what follows we choose the intensity $\lambda > 0$ such that $K - \lambda > -1$. It follows from Lemma 2.2 that the comparison theorem for the BSDE (3.4) holds. Moreover, we also have the following propositions.

Proposition 3.2. *Let the assumption (H5) hold. Then we have, for $s \in [t, T]$, $i = 1, 2$,*

$$\begin{aligned} {}^iY_s^{t,x;u,v} &= {}^{N_s^{t,i}}\tilde{Y}_s^{t,x;u,v}, \\ {}^iH_s^{t,x;u,v} &= m(N_{s-}^{t,i} + 1)\tilde{H}_s^{t,x;u,v} + m(N_{s-}^{t,i})\tilde{Y}_s^{t,x;u,v} - {}^{N_{s-}^{t,i}}\tilde{Y}_{s-}^{t,x;u,v}. \end{aligned}$$

In particular, we have ${}^iY_t^{t,x;u,v} = {}^i\tilde{Y}_t^{t,x;u,v}$, i.e., $J_i(t, x; u, v) = {}^iY_t^{t,x;u,v}$.

Proof. We consider the solution $({}^i\tilde{Y}^{t,x;u,v}, {}^i\tilde{Z}^{t,x;u,v}, {}^i\tilde{H}^{t,x;u,v})$, $i = 1, 2$, of equation (3.3) and, suppressing for simplicity the superscript (t, x, u, v) , we put

$${}^i\hat{Y}_s = {}^{N_s^{t,i}}\tilde{Y}_s, \, {}^i\hat{Z}_s = {}^{N_s^{t,i}}\tilde{Z}_s, \, {}^i\hat{H}_s = m(N_{s-}^{t,i} + 1)\tilde{H}_s + m(N_{s-}^{t,i})\tilde{Y}_{s-} - {}^{N_{s-}^{t,i}}\tilde{Y}_{s-}, \, s \in [t, T].$$

Then, obviously, $({}^i\hat{Y}, {}^i\hat{Z}, {}^i\hat{H}) \in S^2(0, T; \mathbb{R}) \times \mathcal{H}^2(0, T; \mathbb{R}^d) \times \mathcal{H}^2(0, T; \mathbb{R})$. Moreover, setting $\tau_0 = t$ and

$$\tau_l = \inf \left\{ s \geq t, N((t, s]) = l \right\} \wedge T, l \geq 1,$$

we have, on stochastic interval $]]\tau_0, \tau_1[[$,

$$\begin{aligned} -d {}^1\hat{Y}_s &= -d {}^1\tilde{Y}_s = \tilde{f}_1(s, X_s, {}^1\tilde{Y}_s, {}^2\tilde{Y}_s + {}^2\tilde{H}_s, {}^1\tilde{Z}_s, u_s, v_s)ds - {}^1\tilde{Z}_s dB_s \\ &= \tilde{f}_1(s, X_s, {}^1\tilde{Y}_s, ({}^2\tilde{Y}_s - {}^1\tilde{Y}_s + {}^2\tilde{H}_s) + {}^1\tilde{Y}_s, {}^1\tilde{Z}_s, u_s, v_s)ds - {}^1\tilde{Z}_s dB_s \\ &= f_1(s, X_s, {}^1\tilde{Y}_s, {}^2\tilde{Y}_s - {}^1\tilde{Y}_s + {}^2\tilde{H}_s, {}^1\tilde{Z}_s, u_s, v_s)ds - {}^1\tilde{Z}_s dB_s \\ &= f_1(s, X_s, {}^1\hat{Y}_s, {}^1\hat{H}_s, {}^1\hat{Z}_s, u_s, v_s)ds - {}^1\hat{Z}_s dB_s. \end{aligned}$$

On the other hand, analysing the jump hight of ${}^1\hat{Y}$ at τ_1 , we get

$$\begin{aligned} \Delta {}^1\hat{Y}_{\tau_1} &= {}^1\hat{Y}_{\tau_1} - {}^1\hat{Y}_{\tau_1-} = ({}^2\tilde{Y}_{\tau_1} - {}^1\tilde{Y}_{\tau_1-})\Delta N_{\tau_1} \\ &= ({}^2\tilde{H}_{\tau_1} + {}^2\tilde{Y}_{\tau_1-} - {}^1\tilde{Y}_{\tau_1-})\Delta N_{\tau_1} = {}^1\hat{H}_{\tau_1}\Delta N_{\tau_1}. \end{aligned}$$

Consequently, $({}^1\hat{Y}, {}^1\hat{Z}, {}^1\hat{H})$ solves (3.4) over the interval $[[\tau_0, \tau_1]]$, with ${}^1\hat{Y}_{\tau_1} = {}^{N_{\tau_1}^{t,1}}\tilde{Y}_{\tau_1}$. By iterating this argument and arguing in a similar way for $i = 2$, we complete the proof. Indeed, from the uniqueness of BSDE (3.4) it follows that $({}^iY^{t,x;u,v}, {}^iZ^{t,x;u,v}, {}^iH^{t,x;u,v}) = ({}^i\hat{Y}, {}^i\hat{Z}, {}^i\hat{H}), i = 1, 2$. \square

From standard BSDEs estimates we have the following:

Proposition 3.3. *There exists some constant $C > 0$ such that, for all $0 \leq t \leq T$, $x, x' \in \mathbb{R}^n$, $u(\cdot) \in \mathcal{U}$ and $v(\cdot) \in \mathcal{V}$,*

$$|{}^iY_t^{t,x;u,v} - {}^iY_t^{t,x';u,v}| \leq C|x - x'|, \quad |{}^iY_t^{t,x;u,v}| \leq C(1 + |x|), \quad \mathbb{P} - a.s.$$

Let us now introduce subspaces of admissible controls and give the definition of NAD strategies. For later applications this has to be done for games over stochastic intervals. Let σ, τ be two stopping times such that $t \leq \sigma \leq \tau \leq T$.

Definition 3.4. *The space $\mathcal{U}_{\sigma,\tau}$ (resp. $\mathcal{V}_{\sigma,\tau}$) of admissible controls for 1th player (resp., 2th player) over the given stochastic time interval $[[\sigma, \tau]]$ is defined as the space of all processes $\{u_r, \sigma \leq r \leq \tau\}$ (resp., $\{v_r, \sigma \leq r \leq \tau\}$), such that, for $u_0 \in U$, the process $\{u_r 1_{[\sigma,\tau]} + u_0 1_{[\sigma,\tau]^c}\}$ (resp., for $v_0 \in V$, the process $\{v_r 1_{[\sigma,\tau]} + v_0 1_{[\sigma,\tau]^c}\}$) are \mathbb{F} -predictable and take its values in U (resp., V).*

Definition 3.5. *A nonanticipating strategy with delay (NAD strategy) for 1th player over the given stochastic time interval $[[\sigma, \tau]]$ is a measurable mapping $\alpha : \mathcal{V}_{\sigma,\tau} \rightarrow \mathcal{U}_{\sigma,\tau}$, the following properties hold:*

1) α is a nonanticipative strategy, i.e., for every \mathbb{F} -stopping time τ' on Ω with $\sigma \leq \tau' \leq \tau$, and for all $v_1, v_2 \in \mathcal{V}_{\sigma,\tau}$ with $v_1 = v_2$ on $[[\sigma, \tau']]$, it holds $\alpha(v_1) = \alpha(v_2)$ on $[[\sigma, \tau']]$. (The identification of $v_1 = v_2$ and $\alpha(v_1) = \alpha(v_2)$ is in the $\text{dtd}\mathbb{P}$ almost everywhere sense.)

2) α is a nonanticipative strategy with delay, i.e., for all $v \in \mathcal{V}_{\sigma,\tau}$, there exists an increasing sequence of stopping times $\{S_n(v)\}_{n \geq 1}$ with

i) $\sigma = S_0(v) \leq S_1(v) \leq \dots \leq S_n(v) \leq \dots \leq \tau$, ii) $\bigcup_{n \geq 1} \{S_n(v) = \tau\} = \Omega$, \mathbb{P} -a.s., such that, for all $\Gamma \in \mathcal{F}_\sigma$ and for all $n \geq 1$ and $v, v' \in \mathcal{V}_{\sigma,\tau}$, it holds: if $v = v'$ on $[[\sigma, S_{n-1}(v)]] \cap (\Gamma \times [t, T])$, then iii) $S_l(v) = S_l(v')$, on Γ , \mathbb{P} -a.s., $1 \leq l \leq n$,

iv) $\alpha(v) = \alpha(v')$, on $[[\sigma, S_n(v)]] \cap (\Gamma \times [t, T])$.

We denote by all such NAD strategies for 1th player $\mathcal{A}_{\sigma,\tau}$. We can define all NAD strategies $\beta : \mathcal{U}_{\sigma,\tau} \rightarrow \mathcal{V}_{\sigma,\tau}$ for 2th player symmetrically and denote by $\mathcal{B}_{\sigma,\tau}$.

We have the following lemma, which turns out to be useful in what follows. Since the proof of this lemma is similar to that in Lin [16], we omit it here.

Lemma 3.6. *For $(\alpha, \beta) \in \mathcal{A}_{t,T} \times \mathcal{B}_{t,T}$, there exists a unique couple of admissible control processes $(u, v) \in \mathcal{U}_{\sigma,\tau} \times \mathcal{V}_{\sigma,\tau}$ such that $\alpha(v) = u, \beta(u) = v$.*

For $(\alpha, \beta) \in \mathcal{A}_{t,T} \times \mathcal{B}_{t,T}$, it follows from Lemma 3.6 that there exists a unique couple of controls $(u, v) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$ such that $(\alpha(v), \beta(u)) = (u, v)$. In this sense, we define

$$(X^{t,x;\alpha,\beta}, {}^iY^{t,x;\alpha,\beta}, {}^iZ^{t,x;\alpha,\beta}, {}^iH^{t,x;\alpha,\beta}) := (X^{t,x;u,v}, {}^iY^{t,x;u,v}, {}^iZ^{t,x;u,v}, {}^iH^{t,x;u,v}),$$

and $J_i(t, x; \alpha, \beta) := J_i(t, x; u, v)$. This definition allows, in particular, to define the value functions W_i and U_i of the game: For all $(t, x) \in [0, T] \times \mathbb{R}^n$, we put

$$\begin{aligned} W_i(t, x) &:= \text{esssup}_{\alpha \in \mathcal{A}_{t,T}} \text{essinf}_{\beta \in \mathcal{B}_{t,T}} J_i(t, x; \alpha, \beta), \\ U_i(t, x) &:= \text{essinf}_{\beta \in \mathcal{B}_{t,T}} \text{esssup}_{\alpha \in \mathcal{A}_{t,T}} J_i(t, x; \alpha, \beta). \end{aligned}$$

The functions W_i and $U_i, i = 1, 2$, are called the lower and upper value functions, respectively. Observe that, according to the definition of esssup and essinf over a uniformly essentially bounded family of \mathcal{F}_t -measurable random variables, both $W_i(t, x)$ and $U_i(t, x)$ are a priori elements of $L^\infty(\Omega, \mathcal{F}_t, \mathbb{P})$. However, we will prove that they are deterministic.

Proposition 3.7. *Let the assumptions (H4) and (H5) hold. Then, for any $(t, x) \in [0, T] \times \mathbb{R}^n$, we have $W_i(t, x) = \mathbb{E}[W_i(t, x)]$, and $U_i(t, x) = \mathbb{E}[U_i(t, x)]$, \mathbb{P} -a.s., $i = 1, 2$.*

This proposition allows to identify $W_i(t, x), U_i(t, x)$ with the deterministic functions $\mathbb{E}[W_i(t, x)], \mathbb{E}[U_i(t, x)]$, respectively, $i = 1, 2$. The following lemma indicates that W_i is invariant by a large class of transformations on Ω . The proposition follows then by a similar argument as in [9] Lemma 3.2.

Lemma 3.8. *For $(t, x) \in [0, T] \times \mathbb{R}^n$, $\tau : \Omega \rightarrow \Omega$ is an invertible \mathcal{F} - \mathcal{F} measurable transformation such that*

- i) τ and $\tau^{-1} : \Omega \rightarrow \Omega$ are $\mathcal{F}_t - \mathcal{F}_t$ measurable;
- ii) $(B_s - B_t) \circ \tau = B_s - B_t, s \in [t, T], N((t, s]) \circ \tau = N((t, s]), s \in [t, T];$
- iii) the law $\mathbb{P} \circ [\tau]^{-1}$ of τ is equivalent to the underlying probability measure \mathbb{P} ,

then $W_i(t, x) \circ \tau = W_i(t, x), \mathbb{P}$ -a.s.

Even if the formulation of the lemma is the same as in [9], the proof is more difficult here. Indeed, in [9] games of the type "strategy again control" were studied, while we investigate here games of the type "NAD strategy against NAD strategy".

Proof. : We give the proof in four steps:

Step 1: For any $u \in \mathcal{U}_{t,T}, v \in \mathcal{V}_{t,T}, J_i(t, x; u, v) \circ \tau = J_i(t, x; u(\tau), v(\tau)), \mathbb{P}$ -a.s.

The proof follows closely from the arguments in [9] or [10] and is therefore omitted.

Step 2: For $\beta \in \mathcal{B}_{t,T}$, let $\hat{\beta}(u) := \beta(u(\tau^{-1}))(\tau), u \in \mathcal{U}_{t,T}$, and for $\alpha \in \mathcal{A}_{t,T}$, let $\hat{\alpha}(v) := \alpha(v(\tau^{-1}))(\tau), v \in \mathcal{V}_{t,T}$. Then, $\hat{\beta} \in \mathcal{B}_{t,T}$ and $\hat{\alpha} \in \mathcal{A}_{t,T}$.

We only give the proof for $\hat{\beta}$, since we can use a similar argument for $\hat{\alpha}$. From the definition of $\hat{\beta}$ we know that $\hat{\beta}$ maps $\mathcal{U}_{t,T}$ into $\mathcal{V}_{t,T}$.

(1) $\hat{\beta}$ is a nonanticipative strategy. Indeed, let $\sigma : \Omega \rightarrow [t, T]$ be an \mathbb{F} -stopping time and $u_1, u_2 \in \mathcal{U}_{t,T}$ such that $u_1 \equiv u_2$ on $[t, \sigma]$. Since $\tau(\mathcal{F}_s) := \{\tau(A), A \in \mathcal{F}_s\} = \mathcal{F}_s, s \in [t, T]$,

the assumptions i) and ii) imply that $\sigma(\tau^{-1})$ is still an \mathbb{F} -stopping time. Obviously, we have $u_1(\tau^{-1}) \equiv u_2(\tau^{-1})$ on $[[t, \sigma(\tau^{-1})]]$. From $\beta \in \mathcal{B}_{t,T}$ it follows that $\beta(u_1(\tau^{-1})) = \beta(u_2(\tau^{-1}))$ on $[[t, \sigma(\tau^{-1})]]$. Consequently,

$$\widehat{\beta}(u_1) = \beta(u_1(\tau^{-1}))(\tau) = \beta(u_2(\tau^{-1}))(\tau) = \widehat{\beta}(u_2) \text{ on } [[t, \sigma]].$$

(2) $\widehat{\beta}$ is a nonanticipative strategy with delay. Since β is a nonanticipative strategy with delay, we have, for all $u \in \mathcal{U}_{t,T}$, the existence of an increasing sequence of stopping times $\{S'_n(u)\}_{n \geq 1}$ with

$$(a) \ t = S'_0(u) \leq S'_1(u) \leq \dots \leq S'_n(u) \leq \dots \leq T,$$

$$(b) \ \bigcup_{n \geq 1} \{S'_n(u) = T\} = \Omega, \mathbb{P}\text{-a.s.},$$

such that, for all $\Gamma \in \mathcal{F}_t$ and for all $n \geq 1$ and $u, u' \in \mathcal{U}_{t,T}$, it holds: if $u = u'$ on $[[t, S'_{n-1}(u)]] \cap (\Gamma \times [t, T])$, then

$$(c) \ S'_l(u) = S'_l(u'), \text{ on } \Gamma, \mathbb{P}\text{-a.s.}, 1 \leq l \leq n,$$

$$(d) \ \beta(u) = \beta(u'), \text{ on } [t, S'_n(u)] \cap (\Gamma \times [t, T]).$$

For all $u \in \mathcal{U}_{t,T}$, we put $S_n(u) = S'_n(u(\tau^{-1}))(\tau), n \geq 1$. It is easy to check that $\widehat{\beta}$ is a nonanticipative strategy with delay. Moreover, since $\beta(u) = \widehat{\beta}(u(\tau))(\tau^{-1}), u \in \mathcal{U}_{t,T}$, we have that $\{\widehat{\beta} \mid \beta \in \mathcal{B}_{t,T}\} = \mathcal{B}_{t,T}$. Ditto for $\widehat{\alpha}$.

Step 3: For all $\alpha \in \mathcal{A}_{t,T}$ we have:

$$\left(\text{esssup}_{\alpha \in \mathcal{A}_{t,T}} \text{essinf}_{\beta \in \mathcal{B}_{t,T}} J_i(t, x; \alpha, \beta) \right) (\tau) = \text{esssup}_{\alpha \in \mathcal{A}_{t,T}} \text{essinf}_{\beta \in \mathcal{B}_{t,T}} \left(J_i(t, x; \alpha, \beta) (\tau) \right), \mathbb{P}\text{-a.s.}$$

Taking into account the properties of τ , this relation can be proven in the same manner as the corresponding relation in [10], also see [6].

Step 4: We now show that $W_i(t, x)(\tau) = W_i(t, x), \mathbb{P}\text{-a.s.}$

Let $(\alpha, \beta) \in \mathcal{A}_{t,T} \times \mathcal{B}_{t,T}$. Then there exists a unique couple $(u, v) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$ such that $\alpha(v) = u, \beta(u) = v$. Consequently, due to Step 2, $\widehat{\alpha}(v(\tau)) = u(\tau)$ and $\widehat{\beta}(u(\tau)) = v(\tau)$, and, thus, from Step 1,

$$J_i(t, x; \alpha, \beta)(\tau) = J_i(t, x; u, v)(\tau) = J_i(t, x; u(\tau), v(\tau)) = J_i(t, x; \widehat{\alpha}, \widehat{\beta}), \mathbb{P}\text{-a.s.}$$

Thus, since $\{\widehat{\alpha} \mid \alpha \in \mathcal{A}_{t,T}\} = \mathcal{A}_{t,T}$ and $\{\widehat{\beta} \mid \beta \in \mathcal{B}_{t,T}\} = \mathcal{B}_{t,T}$, we conclude with Step 3,

$$\begin{aligned} W_i(t, x)(\tau) &= \text{esssup}_{\alpha \in \mathcal{A}_{t,T}} \text{essinf}_{\beta \in \mathcal{B}_{t,T}} (J_i(t, x; \alpha, \beta)(\tau)) \\ &= \text{esssup}_{\alpha \in \mathcal{A}_{t,T}} \text{essinf}_{\beta \in \mathcal{B}_{t,T}} J_i(t, x; \widehat{\alpha}, \widehat{\beta}) \\ &= \text{esssup}_{\alpha \in \mathcal{A}_{t,T}} \text{essinf}_{\beta \in \mathcal{B}_{t,T}} J_i(t, x; \alpha, \beta) \\ &= W_i(t, x), \mathbb{P}\text{-a.s.} \end{aligned}$$

We get the wished result. \square

By the definition of $W_i(t, x)$ and Proposition 3.3 we can easily get the following properties for our deterministic function W_i . The same proposition holds for U_i .

Proposition 3.9. *Under the assumptions (H4) and (H5), there exists a constant $C > 0$ such that, for all $0 \leq t \leq T, x, x' \in \mathbb{R}^n$,*

$$|W_i(t, x) - W_i(t, x')| \leq C|x - x'|, \quad |W_i(t, x)| \leq C(1 + |x|).$$

We now recall the notion of stochastic backward semigroups, which was first introduced by Peng [21] to study stochastic optimal control problems, and translated later by Buckdahn and Li [10] to SDGs. For any stopping times σ, τ , with $t \leq \sigma \leq \tau \leq T$, and a random variable $\eta \in L^2(\Omega, \mathcal{F}_\tau, \mathbb{P}; \mathbb{R})$, we define

$${}^i G_{\sigma, \tau}^{t, x; u, v}[\eta] := {}^i \widehat{Y}_\sigma^{t, x; u, v},$$

where $({}^i \widehat{Y}_s^{t, x; u, v}, {}^i \widehat{Z}_s^{t, x; u, v}, {}^i \widehat{H}_s^{t, x; u, v})_{t \leq s \leq \tau}$ is the solution of the following BSDE:

$$\begin{cases} -d {}^i \widehat{Y}_s^{t, x; u, v} &= f_{N_s^{t, i}}(s, X_s^{t, x; u, v}, {}^i \widehat{Y}_s^{t, x; u, v}, {}^i \widehat{H}_s^{t, x; u, v}, {}^i \widehat{Z}_s^{t, x; u, v}, u_s, v_s) ds, \quad s \in [t, \tau] \\ &\quad -\lambda {}^i \widehat{H}_s^{t, x; u, v} ds - {}^i \widehat{Z}_s^{t, x; u, v} dB_s - {}^i \widehat{H}_s^{t, x; u, v} d\widehat{N}_s, \\ {}^i \widehat{Y}_\tau^{t, x; u, v} &= \eta, \end{cases}$$

where $X^{t, x; u, v}$ is the solution of SDE (3.1) with $\zeta = x \in \mathbb{R}^n$.

We have the following dynamic programming principle (DPP) over a stochastic interval for our games.

Theorem 3.10. *Let the assumptions (H4) and (H5) hold. Then the following dynamic programming principle holds: For any stopping time τ with $0 \leq t < \tau \leq T$, $x \in \mathbb{R}^n$, $i = 1, 2$,*

$$\begin{aligned} W_i(t, x) &= \text{esssup}_{\alpha \in \mathcal{A}_{t, \tau}} \text{essinf}_{\beta \in \mathcal{B}_{t, \tau}} {}^i G_{t, \tau}^{t, x; \alpha, \beta}[W_{N_\tau^{t, i}}(\tau, X_\tau^{t, x; \alpha, \beta})], \\ U_i(t, x) &= \text{essinf}_{\beta \in \mathcal{B}_{t, \tau}} \text{esssup}_{\alpha \in \mathcal{A}_{t, \tau}} {}^i G_{t, \tau}^{t, x; \alpha, \beta}[U_{N_\tau^{t, i}}(\tau, X_\tau^{t, x; \alpha, \beta})]. \end{aligned}$$

This DPP for stopping times will play a crucial role in Section 4. Since the proof is rather long and technical, we postpone it to Subsection 6.1.

4 Probabilistic interpretation of associated coupled systems of Isaacs equations

The objective of this section is to give a probabilistic interpretation of coupled systems of Isaacs equations. More precisely, we show that the value functions $U = (U_1, U_2)$ and $W = (W_1, W_2)$, introduced in Section 3, are viscosity solutions of the following coupled Isaacs equations:

$$\begin{cases} \frac{\partial}{\partial t} U_i(t, x) + H_i^+(t, x, U_1(t, x), U_2(t, x), DU_i(t, x), D^2 U_i(t, x)) &= 0, (t, x) \in [0, T) \times \mathbb{R}^n, \\ U_i(T, x) &= \Phi_i(x), \quad i = 1, 2, \end{cases} \quad (4.1)$$

and

$$\begin{cases} \frac{\partial}{\partial t} W_i(t, x) + H_i^-(t, x, W_1(t, x), W_2(t, x), DW_i(t, x), D^2 W_i(t, x)) &= 0, (t, x) \in [0, T) \times \mathbb{R}^n, \\ W_i(T, x) &= \Phi_i(x), \quad i = 1, 2, \end{cases} \quad (4.2)$$

respectively, where for $(t, x, y_1, y_2, p, A, u, v) \in [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d \times U \times V$,

$$H_i(t, x, y_1, y_2, p, A, u, v) = \frac{1}{2} \text{tr}(\sigma \sigma^T(t, x, u, v) A) + pb(t, x, u, v) + \widetilde{f}_i(t, x, y_1, y_2, p\sigma(t, x, u, v), u, v),$$

$$\begin{aligned} H_i^-(t, x, y_1, y_2, p, A) &= \sup_{u \in U} \inf_{v \in V} H_i(t, x, y_1, y_2, p, A, u, v), \\ H_i^+(t, x, y_1, y_2, p, A) &= \inf_{v \in V} \sup_{u \in U} H_i(t, x, y_1, y_2, p, A, u, v). \end{aligned}$$

Let us recall the definition of a viscosity solution of the system (4.1). We denote by $C_{l, b}^3([0, T] \times \mathbb{R}^n)$ the set of real-valued functions which are continuously differentiable up to third order and whose derivatives of order from 1 to 3 are bounded.

Definition 4.1. A continuous function $V = (V_1, V_2) \in C([0, T] \times \mathbb{R}^n; \mathbb{R}^2)$ is called

(i) a viscosity subsolution of the system (4.1) if $V_i(T, x) \leq \Phi_i(x)$, for all $i = 1, 2, x \in \mathbb{R}^n$, and if for all test functions $\varphi \in C_{l,b}^3([0, T] \times \mathbb{R}^n)$, $i = 1, 2$, and $(t, x) \in [0, T) \times \mathbb{R}^n$ such that $V_i - \varphi$ attains a local maximum at (t, x) ,

$$\frac{\partial}{\partial t} \varphi(t, x) + H_i^+(t, x, V_1(t, x), V_2(t, x), D\varphi(t, x), D^2\varphi(t, x)) \geq 0, \quad (4.3)$$

(ii) a viscosity supersolution of the system (4.1) if $V_i(T, x) \geq \Phi_i(x)$, for all $i = 1, 2, x \in \mathbb{R}^n$, and if for all functions $\varphi \in C_{l,b}^3([0, T] \times \mathbb{R}^n)$, $i = 1, 2$, and $(t, x) \in [0, T) \times \mathbb{R}^n$ such that $V_i - \varphi$ attains a local minimum at (t, x) ,

$$\frac{\partial}{\partial t} \varphi(t, x) + H_i^+(t, x, V_1(t, x), V_2(t, x), D\varphi(t, x), D^2\varphi(t, x)) \leq 0, \quad (4.4)$$

(iii) a viscosity solution of the system (4.1) if it is both a viscosity subsolution and a supersolution of the system (4.1). In the same way we can define a viscosity solution of the system (4.2).

We now give the main result in this section. We shall give the proof in Section 6.2 since it is rather lengthy.

Theorem 4.2. Let the assumptions (H4) and (H5) hold. Then $U = (U_1, U_2)$ (resp. $W = (W_1, W_2)$) is a viscosity solution of the system (4.1) (resp. (4.2)).

To state a uniqueness theorem for the viscosity solution of the system (4.1), let us first define the following space:

$$\Theta : = \left\{ \varphi \in C([0, T] \times \mathbb{R}^n) : \text{there exists a constant } A > 0 \text{ such that} \right. \\ \left. \lim_{|x| \rightarrow \infty} |\varphi(t, x)| \exp\{-A[\log((|x|^2 + 1)^{\frac{1}{2}})]^2\} = 0, \text{ uniformly in } t \in [0, T] \right\}.$$

Theorem 4.3. Let the assumptions (H4) and (H5) hold. Then there exists at most one viscosity solution $u \in \Theta$ (resp. $v \in \Theta$) of the system (4.1) (resp. (4.2)).

The proof of the Theorem can be adapted from the arguments in Barles, Buckdahn and Pardoux [1] combined with those of Barles and Imbert [2] to our framework. We omit it here.

Remark 4.4. Since $U = (U_1, U_2)$ (resp. $W = (W_1, W_2)$) is a viscosity solution of linear growth, $U = (U_1, U_2)$ (resp. $W = (W_1, W_2)$) is the unique viscosity solution in Θ of the system (4.1) (resp. (4.2)).

An immediate consequence of this remark is that, under **Isaacs condition**:

For all $(t, x, y_1, y_2, p, A, u, v) \in [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d \times U \times V$, $j = 1, 2$, we have

$$\begin{aligned} & \sup_{u \in U} \inf_{v \in V} \left\{ \frac{1}{2} \text{tr}(\sigma \sigma^T(t, x, u, v) A) + pb(t, x, u, v) + f_j(t, x, y_1, y_2, p\sigma(t, x, u, v), u, v) \right\} \\ &= \inf_{v \in V} \sup_{u \in U} \left\{ \frac{1}{2} \text{tr}(\sigma \sigma^T(t, x, u, v) A) + pb(t, x, u, v) + f_j(t, x, y_1, y_2, p\sigma(t, x, u, v), u, v) \right\}, \end{aligned} \quad (4.5)$$

the upper value and lower value functions coincide:

Corollary 4.5. Let Isaacs condition hold. Then we have, for all $(t, x) \in [0, T] \times \mathbb{R}^n$,

$$(U_1(t, x), U_2(t, x)) = (W_1(t, x), W_2(t, x)).$$

5 Nash equilibrium payoffs for nonzero-sum stochastic differential games with coupled cost functionals

The objective of this section is to investigate Nash equilibrium payoffs for nonzero-sum SDGs. An existence theorem and a characterization theorem of Nash equilibrium payoffs are obtained.

In all what follows we assume that the Isaacs condition (4.5) holds, and that all the coefficients are bounded. This latter assumption is not necessary, but has the objective to simplify the arguments. Under the Isaacs condition (4.5), from Corollary 4.5, we have, for $(t, x) \in [0, T] \times \mathbb{R}^n$,

$$\begin{aligned} W_1(t, x) &= \operatorname{esssup}_{\alpha \in \mathcal{A}_{t,T}} \operatorname{essinf}_{\beta \in \mathcal{B}_{t,T}} J_1(t, x; \alpha, \beta) = \operatorname{essinf}_{\beta \in \mathcal{B}_{t,T}} \operatorname{esssup}_{\alpha \in \mathcal{A}_{t,T}} J_1(t, x; \alpha, \beta), \\ W_2(t, x) &= \operatorname{essinf}_{\alpha \in \mathcal{A}_{t,T}} \operatorname{esssup}_{\beta \in \mathcal{B}_{t,T}} J_2(t, x; \alpha, \beta) = \operatorname{esssup}_{\beta \in \mathcal{B}_{t,T}} \operatorname{essinf}_{\alpha \in \mathcal{A}_{t,T}} J_2(t, x; \alpha, \beta). \end{aligned} \quad (5.1)$$

We now give the definition of Nash equilibrium payoffs for nonzero-sum SDGs, which is similar to that in [7] and [16].

Definition 5.1. For $(t, x) \in [0, T] \times \mathbb{R}^n$, a couple $(e_1, e_2) \in \mathbb{R}^2$ is called a Nash equilibrium payoff at the point (t, x) , if for any $\varepsilon > 0$, there exists a couple $(\alpha_\varepsilon, \beta_\varepsilon) \in \mathcal{A}_{t,T} \times \mathcal{B}_{t,T}$ such that, for all $(\alpha, \beta) \in \mathcal{A}_{t,T} \times \mathcal{B}_{t,T}$,

$$J_1(t, x; \alpha_\varepsilon, \beta_\varepsilon) \geq J_1(t, x; \alpha, \beta_\varepsilon) - \varepsilon, \quad J_2(t, x; \alpha_\varepsilon, \beta_\varepsilon) \geq J_2(t, x; \alpha_\varepsilon, \beta) - \varepsilon, \quad \mathbb{P} - a.s., \quad (5.2)$$

and

$$|\mathbb{E}[J_j(t, x; \alpha_\varepsilon, \beta_\varepsilon)] - e_j| \leq \varepsilon, \quad j = 1, 2.$$

Remark 5.2. We notice that unlike in [7] our cost functionals $J_j(t, x; \alpha, \beta)$, $j = 1, 2$, are not necessarily deterministic. Indeed, while [7] is based on the approach by Fleming and Souganidis [11] in which the admissible cost functionals for a game over the fixed time interval $[t, T]$ are independent of \mathcal{F}_t , the present paper is based on the approaches developed in [10], [6] and in [9].

The following equivalent condition of (5.2) follows easily from Lemma 3.6.

Lemma 5.3. For any $\varepsilon > 0$, let $(\alpha_\varepsilon, \beta_\varepsilon) \in \mathcal{A}_{t,T} \times \mathcal{B}_{t,T}$. Then (5.2) holds if and only if, for all $(u, v) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$,

$$J_1(t, x; \alpha_\varepsilon, \beta_\varepsilon) \geq J_1(t, x; u, \beta_\varepsilon(u)) - \varepsilon, \quad J_2(t, x; \alpha_\varepsilon, \beta_\varepsilon) \geq J_2(t, x; \alpha_\varepsilon(v), v) - \varepsilon, \quad \mathbb{P} - a.s. \quad (5.3)$$

As in [16], before giving the characterization theorem of Nash equilibrium payoffs, we first state two important lemmas.

Lemma 5.4. For $(t, x) \in [0, T] \times \mathbb{R}^n$, we fix arbitrarily $u \in \mathcal{U}_{t,T}$. Then,

(i) for all stopping time $\tau \in [t, T]$ and $\varepsilon > 0$, there exists an NAD strategy $\alpha \in \mathcal{A}_{t,T}$ such that, for all $v \in \mathcal{V}_{t,T}$,

$$\begin{aligned} \alpha(v) &= u, \text{ on } [[t, \tau]], \\ {}^2Y_\tau^{t,x;\alpha(v),v} &\leq W_{N_\tau^{t,2}}(\tau, X_\tau^{t,x;\alpha(v),v}) + \varepsilon, \quad \mathbb{P} - a.s. \end{aligned}$$

(ii) for all stopping time $\tau \in [t, T]$ and $\varepsilon > 0$, there exists an NAD strategy $\alpha \in \mathcal{A}_{t,T}$ such that, for all $v \in \mathcal{V}_{t,T}$,

$$\begin{aligned} \alpha(v) &= u, \text{ on } [[t, \tau]], \\ {}^1Y_\tau^{t,x;\alpha(v),v} &\geq W_{N_\tau^{t,1}}(\tau, X_\tau^{t,x;\alpha(v),v}) - \varepsilon, \quad \mathbb{P} - a.s. \end{aligned}$$

Proof. Let us only give the proof of (i); that of (ii) can be carried out in a similar argument. We notice that $\mathcal{V}_{\tau,T}$ can be regarded as a subset of $\mathcal{B}_{\tau,T}$. Indeed, putting $\beta^{v'}(u') = v'$, $u' \in \mathcal{U}_{\tau,T}$, we associate all $v' \in \mathcal{V}_{\tau,T}$ with some $\beta^{v'} \in \mathcal{B}_{\tau,T}$. Therefore, for any $y \in \mathbb{R}^n$, from Proposition 6.13 it follows that

$$\begin{aligned} W_{N_\tau^{t,2}}(\tau, y) &= \operatorname{essinf}_{\alpha \in \mathcal{A}_{\tau,T}} \operatorname{esssup}_{\beta \in \mathcal{B}_{\tau,T}} J_{N_\tau^{t,2}}(\tau, y; \alpha, \beta) \\ &\geq \operatorname{essinf}_{\alpha \in \mathcal{A}_{\tau,T}} \operatorname{esssup}_{v \in \mathcal{V}_{\tau,T}} J_{N_\tau^{t,2}}(\tau, y; \alpha(v), v), \quad \mathbb{P} - a.s. \end{aligned}$$

Then, for any $\varepsilon_0 > 0$, by standard arguments we have the existence of $\alpha_y \in \mathcal{A}_{\tau,T}$ such that

$$W_{N_\tau^{t,2}}(\tau, y) \geq \operatorname{esssup}_{v \in \mathcal{V}_{\tau,T}} J_{N_\tau^{t,2}}(\tau, y; \alpha_y(v), v) - \varepsilon_0, \quad \mathbb{P} - a.s. \quad (5.4)$$

We let $\{O_i\}_{i \geq 1} \subset \mathcal{B}(\mathbb{R}^n)$ be a partition of \mathbb{R}^n such that $\sum_{i \geq 1} O_i = \mathbb{R}^n$, $O_i \neq \emptyset$, and $\operatorname{diam}(O_i) \leq \varepsilon_0$, $i \geq 1$. For $y_i \in O_i$, $i \geq 1$ and $v \in \mathcal{V}_{t,T}$, let us set

$$\alpha(v)_s = \begin{cases} u_s, & s \in [[t, \tau]], \\ \sum_{i \geq 1} 1_{\{X_\tau^{t,x;u,v} \in O_i\}} \alpha_{y_i}(v|_{[\tau,T]})_s, & s \in]\tau, T]. \end{cases} \quad (5.5)$$

Then $\alpha : \mathcal{V}_{t,T} \rightarrow \mathcal{U}_{t,T}$ is an NAD strategy. This can be checked in a straight-forward way and is omitted in order to shorten the proof.

By virtue of Proposition 3.9, (5.4) and (5.5), we deduce that, for $v \in \mathcal{V}_{t,T}$,

$$\begin{aligned} &W_{N_\tau^{t,2}}(\tau, X_\tau^{t,x;\alpha(v),v}) \\ &\geq \sum_{i \geq 1} 1_{\{X_\tau^{t,x;\alpha(v),v} \in O_i\}} W_{N_\tau^{t,2}}(\tau, y_i) - C\varepsilon_0 \\ &\geq \sum_{i \geq 1} 1_{\{X_\tau^{t,x;\alpha(v),v} \in O_i\}} J_{N_\tau^{t,2}}(\tau, y_i; \alpha_{y_i}(v|_{[\tau,T]}), v) - C\varepsilon_0 \\ &= \sum_{i \geq 1} 1_{\{X_\tau^{t,x;\alpha(v),v} \in O_i\}} J_{N_\tau^{t,2}}(\tau, y_i; \alpha(v), v) - C\varepsilon_0. \end{aligned}$$

Consequently, due to Proposition 3.3 we conclude

$$\begin{aligned} W_{N_\tau^{t,2}}(\tau, X_\tau^{t,x;\alpha(v),v}) &\geq \sum_{i \geq 1} 1_{\{X_\tau^{t,x;\alpha(v),v} \in O_i\}} J_{N_\tau^{t,2}}(\tau, X_\tau^{t,x;\alpha(v),v}; \alpha(v), v) - C\varepsilon_0 \\ &= J_{N_\tau^{t,2}}(\tau, X_\tau^{t,x;\alpha(v),v}; \alpha(v), v) - C\varepsilon_0, \end{aligned}$$

where C is a constant which can be different from line to line and is independent of $v \in \mathcal{V}_{t,T}$.

Recalling that $\varepsilon_0 > 0$ hasn't been specified yet, let us choose $\varepsilon_0 = C^{-1}\varepsilon$. We observe that

$$J_{N_\tau^{t,2}}(\tau, X_\tau^{t,x;\alpha(v),v}; \alpha(v), v) = {}^{N_\tau^{t,2}}Y_\tau^{\tau, X_\tau^{t,x;\alpha(v),v}, \alpha(v), v} = {}^2Y_\tau^{t,x,\alpha(v),v}.$$

Then we deduce that $W_{N_\tau^{t,2}}(\tau, X_\tau^{t,x;\alpha(v),v}) \geq {}^2Y_\tau^{t,x,\alpha(v),v} - \varepsilon$, $v \in \mathcal{V}_{t,T}$. This conclude the proof. \square

The following lemma follows from standard estimates for SDEs.

Lemma 5.5. *There exists a positive constant C such that, for all $(u, v), (u', v') \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$, and for all \mathbb{F} -stopping times $\sigma : \Omega \rightarrow [t, T]$ with $X_\sigma^{t,x;u,v} = X_\sigma^{t,x;u',v'}$, $\mathbb{P} - a.s.$, we have the following estimate: for all real $r \in [t, T]$,*

$$\mathbb{E} \left[\sup_{0 \leq s \leq r} |X_{(\sigma+s) \wedge T}^{t,x;u,v} - X_{(\sigma+s) \wedge T}^{t,x;u',v'}|^2 \middle| \mathcal{F}_t \right] \leq Cr, \quad \mathbb{P} - a.s.$$

Let us now give the following characterization theorem of Nash equilibrium payoffs for two-player nonzero-sum SDGs.

Theorem 5.6. *For $(t, x) \in [0, T] \times \mathbb{R}^n$, a couple $(e_1, e_2) \in \mathbb{R}^2$ is a Nash equilibrium payoff at point (t, x) if and only if, for all $\varepsilon > 0$, there exists a couple $(u^\varepsilon, v^\varepsilon) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$ such that for all stopping time $\tau (t \leq \tau \leq T)$ independent of \mathcal{F}_t , and $j = 1, 2$,*

$$\mathbb{P} \left(N_\tau^{t,j} \widetilde{Y}_\tau^{t,x;u^\varepsilon,v^\varepsilon} \geq W_{N_\tau^{t,j}}(\tau, X_\tau^{t,x;u^\varepsilon,v^\varepsilon}) - \varepsilon \mid \mathcal{F}_t \right) \geq 1 - \varepsilon, \quad \mathbb{P} - a.s. \quad (5.6)$$

and

$$|\mathbb{E}[J_j(t, x; u^\varepsilon, v^\varepsilon)] - e_j| \leq \varepsilon. \quad (5.7)$$

Remark 5.7. *The characterization theorem of Nash equilibrium payoffs in [16] generalizes the results in [7] from classical cost functionals without running cost to nonlinear cost functionals with running cost defined by decoupled BSDEs. The above theorem on its part extends the result in [16]: on one hand, we generalize [16] from SDGs without jumps to those with jumps. On the other hand, our cost functionals are defined by a system of two coupled BSDEs.*

Proof of Theorem 5.6: Necessity of (5.6) and (5.7).

Proof. Let us suppose that $(e_1, e_2) \in \mathbb{R}^2$ is a Nash equilibrium payoff at the point (t, x) . Then, by Definition 5.1 we have, for sufficiently small $\varepsilon > 0$, the existence of $(\alpha_\varepsilon, \beta_\varepsilon) \in \mathcal{A}_{t,T} \times \mathcal{B}_{t,T}$ such that, for all $(\alpha, \beta) \in \mathcal{A}_{t,T} \times \mathcal{B}_{t,T}$

$$J_1(t, x; \alpha_\varepsilon, \beta_\varepsilon) \geq J_1(t, x; \alpha, \beta_\varepsilon) - \varepsilon^4, \quad J_2(t, x; \alpha_\varepsilon, \beta_\varepsilon) \geq J_2(t, x; \alpha_\varepsilon, \beta) - \varepsilon^4, \quad \mathbb{P} - a.s., \quad (5.8)$$

$$|\mathbb{E}[J_j(t, x; \alpha_\varepsilon, \beta_\varepsilon)] - e_j| \leq \varepsilon^4, \quad j = 1, 2. \quad (5.9)$$

Since $(\alpha_\varepsilon, \beta_\varepsilon) \in \mathcal{A}_{t,T} \times \mathcal{B}_{t,T}$, it follows from Lemma 3.6 that there exists a unique couple $(u^\varepsilon, v^\varepsilon) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$ such that $\alpha_\varepsilon(v^\varepsilon) = u^\varepsilon, \beta_\varepsilon(u^\varepsilon) = v^\varepsilon$. We notice that from Proposition 3.2, (5.6) is equivalent to

$$\mathbb{P} \left(j Y_\tau^{t,x;u^\varepsilon,v^\varepsilon} \geq W_{N_\tau^{t,j}}(\tau, X_\tau^{t,x;u^\varepsilon,v^\varepsilon}) - \varepsilon \mid \mathcal{F}_t \right) \geq 1 - \varepsilon, \quad \mathbb{P} - a.s.$$

We prove now by contradiction and suppose that (5.6) doesn't hold true. Then, for all $\varepsilon' > 0$, there exists some $\varepsilon \in (0, \varepsilon')$ and some stopping time $\tau \in [t, T]$ such that, for some $j \in \{1, 2\}$, say for $j = 1$,

$$\mathbb{P} \left(\mathbb{P} \left({}^1 Y_\tau^{t,x;u^\varepsilon,v^\varepsilon} < W_{N_\tau^{t,1}}(\tau, X_\tau^{t,x;u^\varepsilon,v^\varepsilon}) - \varepsilon \mid \mathcal{F}_t \right) > \varepsilon \right) > 0. \quad (5.10)$$

Putting

$$A = \left\{ {}^1 Y_\tau^{t,x;u^\varepsilon,v^\varepsilon} < W_{N_\tau^{t,1}}(\tau, X_\tau^{t,x;u^\varepsilon,v^\varepsilon}) - \varepsilon \right\} \in \mathcal{F}_\tau, \quad (5.11)$$

and applying Lemma 5.4 to u^ε and τ , we have the existence of an NAD strategy $\tilde{\alpha} \in \mathcal{A}_{t,T}$ such that, for all $v \in \mathcal{V}_{t,T}$,

$$\begin{aligned} \tilde{\alpha}(v) &= u^\varepsilon, \text{ on } [[t, \tau]], \\ {}^1 Y_\tau^{t,x;\tilde{\alpha}(v),v} &\geq W_{N_\tau^{t,1}}(\tau, X_\tau^{t,x;\tilde{\alpha}(v),v}) - \frac{\varepsilon}{2}, \quad \mathbb{P} - a.s. \end{aligned} \quad (5.12)$$

From Lemma 3.6 we have the existence of a unique couple $(u, v) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$ such that $\tilde{\alpha}(v) = u$, $\beta_\varepsilon(u) = v$. From (5.12) we see that $u = u^\varepsilon$ on $[[t, \tau]]$, and we put

$$\tilde{u} = \begin{cases} u^\varepsilon, & \text{on } ([[t, \tau][\times \Omega) \cup ([\tau, T]] \times A^c), \\ u, & \text{on } [[\tau, T]] \times A. \end{cases}$$

Consequently, from the nonanticipativity of $\beta_\varepsilon \in \mathcal{B}_{t,T}$ we see that $\beta_\varepsilon(\tilde{u}) = \beta_\varepsilon(u^\varepsilon) = v^\varepsilon$ on $[[t, \tau]]$, and for all $s \in [\tau, T]$,

$$\beta_\varepsilon(\tilde{u})_s = \begin{cases} \beta_\varepsilon(u)_s = v_s, & \text{on } A, \\ \beta_\varepsilon(u^\varepsilon)_s = v_s^\varepsilon, & \text{on } A^c. \end{cases}$$

Hence, standard arguments for SDEs and BSDEs yield that

$$\begin{aligned} X^{t,x;\tilde{u},\beta_\varepsilon(\tilde{u})} &= X^{t,x;u^\varepsilon,v^\varepsilon}, \text{ on } [[t, \tau]], \\ X^{t,x;\tilde{u},\beta_\varepsilon(\tilde{u})} &= \begin{cases} X^{t,x;\tilde{\alpha}(v),v}, & \text{on } [[\tau, T]] \times A, \\ X^{t,x;u^\varepsilon,v^\varepsilon}, & \text{on } [[\tau, T]] \times A^c, \end{cases} \end{aligned}$$

as well as

$${}^1Y_\tau^{t,x;\tilde{u},\beta_\varepsilon(\tilde{u})} = \begin{cases} {}^1Y_\tau^{t,x;\tilde{\alpha}(v),v}, & \text{on } A, \\ {}^1Y_\tau^{t,x;u^\varepsilon,v^\varepsilon}, & \text{on } A^c. \end{cases}$$

Thus,

$$\begin{aligned} J_1(t, x; \tilde{u}, \beta_\varepsilon(\tilde{u})) &= {}^1Y_t^{t,x;\tilde{u},\beta_\varepsilon(\tilde{u})} = {}^1G_{t,\tau}^{t,x;\tilde{u},\beta_\varepsilon(\tilde{u})} [{}^1Y_\tau^{t,x;\tilde{u},\beta_\varepsilon(\tilde{u})}] \\ &= {}^1G_{t,\tau}^{t,x;\tilde{u},\beta_\varepsilon(\tilde{u})} [{}^1Y_\tau^{t,x;\tilde{u},\beta_\varepsilon(\tilde{u})} 1_A + {}^1Y_\tau^{t,x;\tilde{u},\beta_\varepsilon(\tilde{u})} 1_{A^c}] \\ &= {}^1G_{t,\tau}^{t,x;\tilde{u},\beta_\varepsilon(\tilde{u})} [{}^1Y_\tau^{t,x;\tilde{\alpha}(v),v} 1_A + {}^1Y_\tau^{t,x;u^\varepsilon,v^\varepsilon} 1_{A^c}]. \end{aligned}$$

By virtue of Lemma 2.2 and (5.12) we conclude

$$\begin{aligned} J_1(t, x; \tilde{u}, \beta_\varepsilon(\tilde{u})) &\geq {}^1G_{t,\tau}^{t,x;\tilde{u},\beta_\varepsilon(\tilde{u})} [(W_{N_\tau^{t,1}}(\tau, X_\tau^{t,x;\tilde{\alpha}(v),v}) - \frac{\varepsilon}{2}) 1_A + {}^1Y_\tau^{t,x;u^\varepsilon,v^\varepsilon} 1_{A^c}] \\ &= {}^1G_{t,\tau}^{t,x;\tilde{u},\beta_\varepsilon(\tilde{u})} [W_{N_\tau^{t,1}}(\tau, X_\tau^{t,x;u^\varepsilon,v^\varepsilon}) 1_A + {}^1Y_\tau^{t,x;u^\varepsilon,v^\varepsilon} 1_{A^c} - \frac{\varepsilon}{2} 1_A]. \end{aligned}$$

Therefore, from (5.11) we deduce that

$$\begin{aligned} J_1(t, x; \tilde{u}, \beta_\varepsilon(\tilde{u})) &\geq {}^1G_{t,t+\delta}^{t,x;\tilde{u},\beta_\varepsilon(\tilde{u})} [({}^1Y_\tau^{t,x;u^\varepsilon,v^\varepsilon} + \varepsilon) 1_A + {}^1Y_\tau^{t,x;u^\varepsilon,v^\varepsilon} 1_{A^c} - \frac{\varepsilon}{2} 1_A] \\ &= {}^1G_{t,t+\delta}^{t,x;\tilde{u},\beta_\varepsilon(\tilde{u})} [{}^1Y_\tau^{t,x;u^\varepsilon,v^\varepsilon} + \frac{\varepsilon}{2} 1_A] \\ &= {}^1G_{t,\tau}^{t,x;u^\varepsilon,v^\varepsilon} [{}^1Y_\tau^{t,x;u^\varepsilon,v^\varepsilon} + \frac{\varepsilon}{2} 1_A]. \end{aligned} \tag{5.13}$$

Putting

$$y_s = {}^1G_{s,\tau}^{t,x;u^\varepsilon,v^\varepsilon} [{}^1Y_\tau^{t,x;u^\varepsilon,v^\varepsilon} + \frac{\varepsilon}{2} 1_A], \quad s \in [t, \tau],$$

let us consider the following BSDE:

$$\begin{aligned} y_s &= {}^1Y_\tau^{t,x;u^\varepsilon,v^\varepsilon} + \frac{\varepsilon}{2} 1_A + \int_s^\tau f_{N_r^{t,1}}(r, X_r^{t,x;u^\varepsilon,v^\varepsilon}, y_r, h_r, z_r, u_r^\varepsilon, v_r^\varepsilon) dr \\ &\quad - \lambda \int_s^\tau h_r dr - \int_s^\tau z_r dB_r - \int_s^\tau h_r d\tilde{N}_r, \quad s \in [t, \tau], \end{aligned}$$

as well as

$${}^1Y_s^{t,x;u^\varepsilon,v^\varepsilon} = {}^1Y_\tau^{t,x;u^\varepsilon,v^\varepsilon} + \int_s^\tau f_{N_r^{t,1}}(r, X_r^{t,x;u^\varepsilon,v^\varepsilon}, {}^1Y_r^{t,x;u^\varepsilon,v^\varepsilon}, {}^1H_r^{t,x;u^\varepsilon,v^\varepsilon}, {}^1Z_r^{t,x;u^\varepsilon,v^\varepsilon}, u_r^\varepsilon, v_r^\varepsilon) dr$$

$$-\lambda \int_s^\tau {}^1H_r^{t,x;u^\varepsilon,v^\varepsilon} dr - \int_s^\tau {}^1Z_r^{t,x;u^\varepsilon,v^\varepsilon} dB_r - \int_s^\tau {}^1H_r^{t,x;u^\varepsilon,v^\varepsilon} d\tilde{N}_r, \quad s \in [t, \tau].$$

In order to simplify the notations we suppose until the end of this proof that the dimension of the Brownian motion d is equal to 1, since we can use a similar arguments for $d > 1$. Let us set

$$\bar{y}_s = y_s - {}^1Y_s^{t,x;u^\varepsilon,v^\varepsilon}, \quad \bar{h}_s = h_s - {}^1H_s^{t,x;u^\varepsilon,v^\varepsilon}, \quad \bar{z}_s = z_s - {}^1Z_s^{t,x;u^\varepsilon,v^\varepsilon}, \quad s \in [t, \tau].$$

Then we conclude

$$\begin{aligned} \bar{y}_s &= \frac{\varepsilon}{2} 1_A + \int_s^\tau [f_{N_r^{t,1}}(r, X_r^{t,x;u^\varepsilon,v^\varepsilon}, y_r, h_r, z_r, u_r^\varepsilon, v_r^\varepsilon) \\ &\quad - f_{N_r^{t,1}}(r, X_r^{t,x;u^\varepsilon,v^\varepsilon}, {}^1Y_r^{t,x;u^\varepsilon,v^\varepsilon}, {}^1H_r^{t,x;u^\varepsilon,v^\varepsilon}, {}^1Z_r^{t,x;u^\varepsilon,v^\varepsilon}, u_r^\varepsilon, v_r^\varepsilon)] dr \\ &\quad - \lambda \int_s^\tau \bar{h}_r dr - \int_s^\tau \bar{z}_r dB_r - \int_s^\tau \bar{h}_r d\tilde{N}_r, \quad s \in [t, \tau]. \end{aligned} \quad (5.14)$$

Let us put, for $r \in [t, \tau]$,

$$\begin{aligned} a_r &= 1_{\{\bar{y}_r \neq 0\}} (\bar{y}_r)^{-1} \left(f_{N_r^{t,1}}(r, X_r^{t,x;u^\varepsilon,v^\varepsilon}, y_r, h_r, z_r, u_r^\varepsilon, v_r^\varepsilon) \right. \\ &\quad \left. - f_{N_r^{t,1}}(r, X_r^{t,x;u^\varepsilon,v^\varepsilon}, {}^1Y_r^{t,x;u^\varepsilon,v^\varepsilon}, {}^1H_r^{t,x;u^\varepsilon,v^\varepsilon}, {}^1Z_r^{t,x;u^\varepsilon,v^\varepsilon}, u_r^\varepsilon, v_r^\varepsilon) \right), \\ b_r &= 1_{\{\bar{z}_r \neq 0\}} (\bar{z}_r)^{-1} \left(f_{N_r^{t,1}}(r, X_r^{t,x;u^\varepsilon,v^\varepsilon}, {}^1Y_r^{t,x;u^\varepsilon,v^\varepsilon}, {}^1H_r^{t,x;u^\varepsilon,v^\varepsilon}, z_r, u_r^\varepsilon, v_r^\varepsilon) \right. \\ &\quad \left. - f_{N_r^{t,1}}(r, X_r^{t,x;u^\varepsilon,v^\varepsilon}, {}^1Y_r^{t,x;u^\varepsilon,v^\varepsilon}, {}^1H_r^{t,x;u^\varepsilon,v^\varepsilon}, {}^1Z_r^{t,x;u^\varepsilon,v^\varepsilon}, u_r^\varepsilon, v_r^\varepsilon) \right), \\ c_r &= 1_{\{\bar{h}_r \neq 0\}} (\bar{h}_r)^{-1} \left(f_{N_r^{t,1}}(r, X_r^{t,x;u^\varepsilon,v^\varepsilon}, {}^1Y_r^{t,x;u^\varepsilon,v^\varepsilon}, h_r, z_r, u_r^\varepsilon, v_r^\varepsilon) \right. \\ &\quad \left. - f_{N_r^{t,1}}(r, X_r^{t,x;u^\varepsilon,v^\varepsilon}, {}^1Y_r^{t,x;u^\varepsilon,v^\varepsilon}, {}^1H_r^{t,x;u^\varepsilon,v^\varepsilon}, z_r, u_r^\varepsilon, v_r^\varepsilon) \right). \end{aligned}$$

Then, from (H5) we deduce that $|a_r| \leq C$, $|b_r| \leq C$, $|c_r| \leq C$, and $\hat{c}_r := c_r - \lambda \geq K - \lambda > -1$, $r \in [t, \tau]$. Consequently, there exists a constant $\varepsilon_0 > 0$ small enough, such that $\hat{c}_r \geq -1 + \varepsilon_0$, $r \in [t, \tau]$, and BSDE (5.14) can be written as follows:

$$\bar{y}_s = \frac{\varepsilon}{2} 1_A + \int_s^\tau [a_r \bar{y}_r + b_r \bar{z}_r + \hat{c}_r \bar{h}_r] dr - \int_s^\tau \bar{z}_r dB_r - \int_s^\tau \bar{h}_r d\tilde{N}_r, \quad s \in [t, \tau]. \quad (5.15)$$

Let us put

$$M_{t,\tau} = \exp \left(\int_t^\tau b_r dB_r - \frac{1}{2} \int_t^\tau |b_r|^2 dr - \lambda \int_t^\tau \hat{c}_r dr \right) \prod_{t < r \leq \tau} (1 + \hat{c}_r \Delta N_r).$$

Then from the Girsanov theorem we know that there exists a probability measure $\mathbb{Q} = M_{t,\tau} \cdot \mathbb{P}$ defined on (Ω, \mathcal{F}) such that

$$\hat{N}_s = \tilde{N}_s - \int_t^{s \wedge \tau} \hat{c}_r dr, \quad s \in [t, T],$$

is an (\mathbb{F}, \mathbb{Q}) -martingale, and

$$\hat{B}_s = B_s - \int_t^{s \wedge \tau} b_r dr, \quad s \in [t, T],$$

is an (\mathbb{F}, \mathbb{Q}) -Brownian motion, and both are independent under \mathbb{Q} . Therefore, (5.15) takes the following form:

$$\bar{y}_s = \frac{\varepsilon}{2} 1_A + \int_s^\tau a_r \bar{y}_r dr - \int_s^\tau \bar{z}_r d\hat{B}_r - \int_s^\tau \bar{h}_r d\hat{N}_r, \quad s \in [t, \tau].$$

By applying Itô's formula to $\bar{y}_s e^{\int_t^s a_r dr}$ we obtain

$$\bar{y}_t = \frac{\varepsilon}{2} \mathbb{E}^{\mathbb{Q}}[1_A e^{\int_t^\tau a_r dr} | \mathcal{F}_t] = \frac{\varepsilon}{2} \mathbb{E}[1_A M_{t,\tau} e^{\int_t^\tau a_r dr} | \mathcal{F}_t].$$

Thanks to the Hölder inequality we have

$$\mathbb{P}(A | \mathcal{F}_t)^2 = (\mathbb{E}[1_A | \mathcal{F}_t])^2 \leq \mathbb{E}[1_A M_{t,\tau} e^{\int_t^\tau a_r dr} | \mathcal{F}_t] \mathbb{E}[M_{t,\tau}^{-1} e^{-\int_t^\tau a_r dr} | \mathcal{F}_t].$$

On the other hand,

$$\begin{aligned} & \mathbb{E}[M_{t,t+\delta}^{-1} e^{-\int_t^\tau a_r dr} | \mathcal{F}_t] \\ = & \mathbb{E}[\exp\left(-\int_t^\tau a_r dr - \int_t^\tau b_r dB_r + \frac{1}{2} \int_t^\tau |b_r|^2 dr + \lambda \int_t^\tau \hat{c}_r dr\right) \prod_{t < r \leq \tau} (1 + \hat{c}_r \Delta N_r)^{-1} | \mathcal{F}_t] \\ = & \mathbb{E}[\exp\left(-\int_t^\tau a_r dr - \int_t^\tau b_r dB_r + \frac{1}{2} \int_t^\tau |b_r|^2 dr + \lambda \int_t^\tau \hat{c}_r dr\right) \\ & \quad \times \exp\left(-\int_t^\tau \ln(1 + \hat{c}_r) dN_r\right) | \mathcal{F}_t] \\ \leq & \exp((1 + \lambda)CT) \mathbb{E}[\exp\left(-\int_t^\tau b_r dB_r - \frac{1}{2} \int_t^\tau |b_r|^2 dr\right) \exp\left(-\int_t^\tau \ln(\varepsilon_0) dN_r\right) | \mathcal{F}_t] \\ \leq & C_0^{-1}, \end{aligned}$$

for some suitably chosen constant $C_0 > 0$. Consequently, putting

$$\Delta = \left\{ \mathbb{P}\left({}^1 Y_\tau^{t,x;u^\varepsilon,v^\varepsilon} < W_{N_\tau^{t,1}}(\tau, X_\tau^{t,x;u^\varepsilon,v^\varepsilon}) - \varepsilon \mid \mathcal{F}_t\right) > \varepsilon \right\} \left(= \{\mathbb{P}(A | \mathcal{F}_t) > \varepsilon\} \right),$$

we have

$$\begin{aligned} \bar{y}_t &= \frac{\varepsilon}{2} \mathbb{E}[1_A M_{t,\tau} e^{\int_t^\tau a_r dr} | \mathcal{F}_t] \geq \frac{\varepsilon C_0}{2} (\mathbb{E}[1_A | \mathcal{F}_t])^2 \\ &= \frac{\varepsilon C_0}{2} (\mathbb{P}(A | \mathcal{F}_t))^2 > \frac{\varepsilon^3}{2} C_0 1_\Delta. \end{aligned}$$

Thus, since $\bar{y}_t = y_t - {}^1 Y_t^{t,x;u^\varepsilon,v^\varepsilon} = {}^1 G_{t,\tau}^{t,x;u^\varepsilon,v^\varepsilon} [{}^1 Y_\tau^{t,x;u^\varepsilon,v^\varepsilon} + \frac{\varepsilon}{2} 1_A] - {}^1 G_{t,\tau}^{t,x;u^\varepsilon,v^\varepsilon} [{}^1 Y_\tau^{t,x;u^\varepsilon,v^\varepsilon}]$, we obtain

$${}^1 G_{t,\tau}^{t,x;u^\varepsilon,v^\varepsilon} [{}^1 Y_\tau^{t,x;u^\varepsilon,v^\varepsilon} + \frac{\varepsilon}{2} 1_A] > {}^1 G_{t,\tau}^{t,x;u^\varepsilon,v^\varepsilon} [{}^1 Y_\tau^{t,x;u^\varepsilon,v^\varepsilon}] + \frac{\varepsilon^3}{2} C_0 1_\Delta.$$

Therefore, (5.13) yields $J_1(t, x; \tilde{u}, \beta_\varepsilon(\tilde{u})) > J_1(t, x; \alpha_\varepsilon, \beta_\varepsilon) + \frac{\varepsilon^3}{2} C_0 1_\Delta$. Let us choose ε' sufficiently small such that $\frac{\varepsilon'^3}{2} C_0 > \varepsilon'^4$. Then we have $\frac{\varepsilon^3}{2} C_0 > \varepsilon^4$. Since $\mathbb{P}(\Delta) > 0$, we have a contradiction with (5.8) for $\alpha(\cdot) = \tilde{u}$. The proof is complete. \square

Proof of Theorem 5.6: Sufficiency of (5.6) and (5.7).

Proof. We fix arbitrarily $\varepsilon > 0$. For $\varepsilon_0 > 0$ being specified later let us assume that $(u^{\varepsilon_0}, v^{\varepsilon_0}) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$ satisfies (5.6) and (5.7). In particular, for all $s \in [t, T]$ and $j = 1, 2$,

$$\mathbb{P}\left({}^j Y_s^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}} \geq W_{N_s^{t,j}}(s, X_s^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}}) - \varepsilon_0 \mid \mathcal{F}_t\right) \geq 1 - \varepsilon_0, \quad \mathbb{P} - a.s., \quad (5.16)$$

$$|\mathbb{E}[J_j(t, x; u^{\varepsilon_0}, v^{\varepsilon_0})] - e_j| \leq \varepsilon_0, \quad (5.17)$$

where we use Proposition 3.2 for the first inequality.

We fix arbitrarily a partition: $t = t_0 \leq t_1 \leq \dots \leq t_m = T$ of $[t, T]$, and we put $\delta = \sup_i |t_i - t_{i+1}|$. Let us apply Lemma 5.4 to u^{ε_0} and $\tau = t_1, \dots, t_m$, successively. Then, for $\varepsilon_1 > 0$ (ε_1 depends on ε and is specified later) we have the existence of NAD strategies $\alpha_i \in \mathcal{A}_{t,T}$, $i = 1, \dots, m$, such that, for all $v \in \mathcal{V}_{t,T}$,

$$\begin{aligned} \alpha_i(v) &= u^{\varepsilon_0}, \text{ on } [t, t_i], \\ {}^2Y_{t_i}^{t,x;\alpha_i(v),v} &\leq W_{N_{t_i}^{t,2}}(t_i, X_{t_i}^{t,x;\alpha_i(v),v}) + \varepsilon_1, \mathbb{P} - a.s. \end{aligned} \quad (5.18)$$

For all $v \in \mathcal{V}_{t,T}$, let us define

$$\begin{aligned} S^v &= \inf \left\{ s \geq t \mid \lambda(\{r \in [t, s] : v_r \neq v_r^{\varepsilon_0}\}) > 0 \right\}, \\ t^v &= \inf \left\{ t_i \geq S^v \mid i = 1, \dots, m \right\} \wedge T. \end{aligned}$$

Here we denote by λ the Lebesgue measure on the real line \mathbb{R} . Then, S^v and t^v are stopping times, and $S^v \leq t^v \leq S^v + \delta$. Let us set

$$\alpha_\varepsilon(v) = \begin{cases} u^{\varepsilon_0}, & \text{on } [t, t^v], \\ \alpha_i(v), & \text{on } (t_i, T] \times \{t^v = t_i\}, 1 \leq i \leq m. \end{cases}$$

Then α_ε is an NAD strategy, and by virtue of (5.18) we obtain

$$\begin{aligned} {}^2Y_{t^v}^{t,x;\alpha_\varepsilon(v),v} &= \sum_{i=1}^m {}^2Y_{t_i}^{t,x;\alpha_\varepsilon(v),v} 1_{\{t^v=t_i\}} \\ &\leq \sum_{i=1}^m W_{N_{t_i}^{t,2}}(t_i, X_{t_i}^{t,x;\alpha_\varepsilon(v),v}) 1_{\{t^v=t_i\}} + \varepsilon_1 \\ &= W_{N_{t^v}^{t,2}}(t^v, X_{t^v}^{t,x;\alpha_\varepsilon(v),v}) + \varepsilon_1, \mathbb{P} - a.s. \end{aligned} \quad (5.19)$$

Let us admit the following lemma for the moment; we shall prove it after.

Lemma 5.8. *For all $\varepsilon > 0$ and $v \in \mathcal{V}_{t,T}$,*

$$J_2(t, x; \alpha_\varepsilon(v), v) \leq J_2(t, x; u^{\varepsilon_0}, v^{\varepsilon_0}) + \varepsilon, \quad \alpha_\varepsilon(v^{\varepsilon_0}) = u^{\varepsilon_0}.$$

By a similar argument as that for Lemma 5.8 we can construct $\beta_\varepsilon \in \mathcal{B}_{t,T}$ such that, for all $u \in \mathcal{U}_{t,T}$,

$$J_1(t, x; u, \beta_\varepsilon(u)) \leq J_1(t, x; u^{\varepsilon_0}, v^{\varepsilon_0}) + \varepsilon, \quad \beta_\varepsilon(u^{\varepsilon_0}) = v^{\varepsilon_0}.$$

From the latter both inequalities, (5.17) and Lemma 5.3 it follows that $(\alpha_\varepsilon, \beta_\varepsilon)$ satisfies Definition 5.1. Consequently, (e_1, e_2) is a Nash equilibrium payoff. \square

Proof of Lemma 5.8: From (5.19), the Lemmas 2.2 and 2.3 we get the existence of a positive constant C such that

$$\begin{aligned} J_2(t, x, \alpha_\varepsilon(v), v) &= {}^2G_{t,t^v}^{t,x;\alpha_\varepsilon(v),v} [{}^2Y_{t^v}^{t,x;\alpha_\varepsilon(v),v}] \\ &\leq {}^2G_{t,t^v}^{t,x;\alpha_\varepsilon(v),v} [W_{N_{t^v}^{t,2}}(t^v, X_{t^v}^{t,x;\alpha_\varepsilon(v),v}) + \varepsilon_1] \\ &\leq {}^2G_{t,t^v}^{t,x;\alpha_\varepsilon(v),v} [W_{N_{t^v}^{t,2}}(t^v, X_{t^v}^{t,x;\alpha_\varepsilon(v),v})] + C\varepsilon_1. \end{aligned} \quad (5.20)$$

From Proposition 3.9, Lemma 5.5 as well as the definitions of t^v and α_ε it follows that

$$\mathbb{E}[|W_{N_{t^v}^{t,2}}(t^v, X_{t^v}^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}}) - W_{N_{t^v}^{t,2}}(t^v, X_{t^v}^{t,x;\alpha_\varepsilon(v),v})|^2 | \mathcal{F}_t] \leq C\delta, \quad \mathbb{P} - a.s.$$

Consequently, by virtue of Lemma 2.3 we deduce that

$$\begin{aligned} & | {}^2G_{t,t^v}^{t,x;\alpha_\varepsilon(v),v}[W_{N_{t^v}^{t,2}}(t^v, X_{t^v}^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}})] - {}^2G_{t,t^v}^{t,x;\alpha_\varepsilon(v),v}[W_{N_{t^v}^{t,2}}(t^v, X_{t^v}^{t,x;\alpha_\varepsilon(v),v})] | \\ & \leq C\mathbb{E}[|W_{N_{t^v}^{t,2}}(t^v, X_{t^v}^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}}) - W_{N_{t^v}^{t,2}}(t^v, X_{t^v}^{t,x;\alpha_\varepsilon(v),v})|^2 | \mathcal{F}_t]^{\frac{1}{2}} \leq C\delta^{\frac{1}{2}}, \end{aligned}$$

from which, combined with (5.20), we get

$$\begin{aligned} & J_2(t, x, \alpha_\varepsilon(v), v) \\ & \leq {}^2G_{t,t^v}^{t,x;\alpha_\varepsilon(v),v}[W_{N_{t^v}^{t,2}}(t^v, X_{t^v}^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}})] + C\varepsilon_1 \\ & \quad + | {}^2G_{t,t^v}^{t,x;\alpha_\varepsilon(v),v}[W_{N_{t^v}^{t,2}}(t^v, X_{t^v}^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}})] - {}^2G_{t,t^v}^{t,x;\alpha_\varepsilon(v),v}[W_{N_{t^v}^{t,2}}(t^v, X_{t^v}^{t,x;\alpha_\varepsilon(v),v})] | \\ & \leq {}^2G_{t,t^v}^{t,x;\alpha_\varepsilon(v),v}[W_{N_{t^v}^{t,2}}(t^v, X_{t^v}^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}})] + C\varepsilon_1 + C\delta^{\frac{1}{2}}. \end{aligned}$$

For $s \in [t, T]$, we put

$$\Omega_s = \left\{ {}^2Y_s^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}} \geq W_{N_s^{t,2}}(s, X_s^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}}) - \varepsilon_0 \right\}.$$

Then we have

$$\begin{aligned} J_2(t, x; \alpha_\varepsilon(v), v) & \leq {}^2G_{t,t^v}^{t,x;\alpha_\varepsilon(v),v}[W_{N_{t^v}^{t,2}}(t^v, X_{t^v}^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}})] + C\varepsilon_1 + C\tau^{\frac{1}{2}} \\ & \leq {}^2G_{t,t^v}^{t,x;\alpha_\varepsilon(v),v}\left[\sum_{i=1}^m W_{N_{t_i}^{t,2}}(t_i, X_{t_i}^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}})1_{\{t^v=t_i\}}1_{\Omega_{t_i}}\right] + C\varepsilon_1 + C\tau^{\frac{1}{2}} + I, \end{aligned} \quad (5.21)$$

where

$$\begin{aligned} I & = | {}^2G_{t,t^v}^{t,x;\alpha_\varepsilon(v),v}\left[\sum_{i=1}^m W_{N_{t_i}^{t,2}}(t_i, X_{t_i}^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}})1_{\{t^v=t_i\}}\right] \\ & \quad - {}^2G_{t,t^v}^{t,x;\alpha_\varepsilon(v),v}\left[\sum_{i=1}^m W_{N_{t_i}^{t,2}}(t_i, X_{t_i}^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}})1_{\{t^v=t_i\}}1_{\Omega_{t_i}}\right] |. \end{aligned}$$

Noting Lemma 2.3, (5.16) as well as the boundedness of $W_{N_{t_i}^{t,2}}$ we conclude

$$I \leq \mathbb{E}\left[\sum_{i=1}^m |W_{N_{t_i}^{t,2}}(t_i, X_{t_i}^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}})|^2 1_{\{t^v=t_i\}}1_{\Omega_{t_i}^c} \Big| \mathcal{F}_t\right]^{\frac{1}{2}} \leq C \sum_{i=1}^m \mathbb{P}(\Omega_{t_i}^c | \mathcal{F}_t)^{\frac{1}{2}} \leq Cm\varepsilon_0^{\frac{1}{2}}. \quad (5.22)$$

Since ${}^2Y_{t_i}^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}} \geq W_{N_{t_i}^{t,2}}(t_i, X_{t_i}^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}}) - \varepsilon_0$ on Ω_{t_i} , it follows from Lemma 2.3 that

$$\begin{aligned} & {}^2G_{t,t^v}^{t,x;\alpha_\varepsilon(v),v}\left[\sum_{i=1}^m W_{N_{t_i}^{t,2}}(t_i, X_{t_i}^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}})1_{\{t^v=t_i\}}1_{\Omega_{t_i}}\right] \\ & \leq {}^2G_{t,t^v}^{t,x;\alpha_\varepsilon(v),v}\left[\sum_{i=1}^m {}^2Y_{t_i}^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}}1_{\{t^v=t_i\}}1_{\Omega_{t_i}} + \varepsilon_0\right] \\ & \leq {}^2G_{t,t^v}^{t,x;\alpha_\varepsilon(v),v}\left[\sum_{i=1}^m {}^2Y_{t_i}^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}}1_{\{t^v=t_i\}}1_{\Omega_{t_i}}\right] + C\varepsilon_0. \end{aligned}$$

Similarly to (5.22) we have

$$| {}^2G_{t,t^v}^{t,x;\alpha_\varepsilon(v),v} [\sum_{i=1}^m {}^2Y_{t_i}^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}} 1_{\{t^v=t_i\}} 1_{\Omega_{t_i}}] - {}^2G_{t,t^v}^{t,x;\alpha_\varepsilon(v),v} [\sum_{i=1}^m {}^2Y_{t_i}^{t,x;\alpha_\varepsilon(v),v} 1_{\{t^v=t_i\}}] | \leq Cm\varepsilon_0^{\frac{1}{2}},$$

from which together with ${}^2Y_{t^v}^{t,x;\alpha_\varepsilon(v),v} = \sum_{i=1}^m {}^2Y_{t_i}^{t,x;\alpha_\varepsilon(v),v} 1_{\{t^v=t_i\}}$ it follows that

$$\begin{aligned} & {}^2G_{t,t^v}^{t,x;\alpha_\varepsilon(v),v} [\sum_{i=1}^m W_{N_{t_i}^{t,2}}(t_i, X_{t_i}^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}}) 1_{\{t^v=t_i\}} 1_{\Omega_{t_i}}] \\ & \leq {}^2G_{t,t^v}^{t,x;\alpha_\varepsilon(v),v} [{}^2Y_{t^v}^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}}] + C\varepsilon_0 + Cm\varepsilon_0^{\frac{1}{2}} \\ & \leq | {}^2G_{t,t^v}^{t,x;\alpha_\varepsilon(v),v} [{}^2Y_{t^v}^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}}] - {}^2G_{t,t^v}^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}} [{}^2Y_{t^v}^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}}] | \\ & \quad + {}^2G_{t,t^v}^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}} [{}^2Y_{t^v}^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}}] + C\varepsilon_0 + Cm\varepsilon_0^{\frac{1}{2}} \\ & = | {}^2G_{t,t^v}^{t,x;\alpha_\varepsilon(v),v} [{}^2Y_{t^v}^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}}] - {}^2G_{t,t^v}^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}} [{}^2Y_{t^v}^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}}] | \\ & \quad + J_2(t, x; u^{\varepsilon_0}, v^{\varepsilon_0}) + C\varepsilon_0 + Cm\varepsilon_0^{\frac{1}{2}}. \end{aligned}$$

Using arguments similar to those in [16] we can show that

$$| {}^2G_{t,t^v}^{t,x;\alpha_\varepsilon(v),v} [{}^2Y_{t^v}^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}}] - {}^2G_{t,t^v}^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}} [{}^2Y_{t^v}^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}}] | \leq C\delta^{\frac{1}{2}}.$$

Consequently,

$${}^2G_{t,t^v}^{t,x;\alpha_\varepsilon(v),v} [\sum_{i=1}^m W_{N_{t_i}^{t,2}}(t_i, X_{t_i}^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}}) 1_{\{t^v=t_i\}} 1_{\Omega_{t_i}}] \leq C\delta^{\frac{1}{2}} + J_2(t, x; u^{\varepsilon_0}, v^{\varepsilon_0}) + C\varepsilon_0 + Cm\varepsilon_0^{\frac{1}{2}}.$$

Thus, from (5.21) and (5.22) we have

$$J_2(t, x; \alpha_\varepsilon(v), v) \leq J_2(t, x; u^{\varepsilon_0}, v^{\varepsilon_0}) + C\varepsilon_0 + Cm\varepsilon_0^{\frac{1}{2}} + C\varepsilon_1 + C\delta^{\frac{1}{2}}.$$

We can choose $\tau > 0, \varepsilon_0 > 0$, and $\varepsilon_1 > 0$ such that $C\varepsilon_0 + Cm\varepsilon_0^{\frac{1}{2}} + C\varepsilon_1 + C\delta^{\frac{1}{2}} \leq \varepsilon$ and $\varepsilon_0 < \varepsilon$. Thus,

$$J_2(t, x; \alpha_\varepsilon(v), v) \leq J_2(t, x; u^{\varepsilon_0}, v^{\varepsilon_0}) + \varepsilon, v \in \mathcal{V}_{t,T}.$$

This allows us to complete the proof. \square

One of our main results of this section is the following existence theorem of a Nash equilibrium payoff.

Theorem 5.9. *Under the Isaacs condition, there exists a Nash equilibrium payoff at (t, x) , for all $(t, x) \in [0, T] \times \mathbb{R}^n$.*

From Theorem 5.6 we only have to prove that, for all $\varepsilon > 0$, there exists $(u^\varepsilon, v^\varepsilon) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$ such that (5.6) and (5.7) hold, for $s \in [t, T], j = 1, 2$. The following proposition is crucial for this proof and it will be proven after.

Proposition 5.10. *Let the assumptions of Theorem 5.6 hold. Then, for all $\varepsilon > 0$, there exists $(u^\varepsilon, v^\varepsilon) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$ independent of \mathcal{F}_t such that, for all stopping time τ ($t \leq \tau \leq T$) independent of \mathcal{F}_t , $j = 1, 2$, (5.6) holds:*

$$\mathbb{P} \left(N_\tau^{t,j} \tilde{Y}_\tau^{t,x;u^\varepsilon,v^\varepsilon} \geq W_{N_\tau^{t,j}}(\tau, X_\tau^{t,x;u^\varepsilon,v^\varepsilon}) - \varepsilon \mid \mathcal{F}_t \right) \geq 1 - \varepsilon, \mathbb{P} - a.s.$$

We now give the proof of Theorem 5.9.

Proof. For $\varepsilon > 0$, let $(u^\varepsilon, v^\varepsilon) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$ be obtained by Proposition 5.10. Then (5.6) holds. By noticing that $(u^\varepsilon, v^\varepsilon)$ is independent of \mathcal{F}_t we see that $J_j(t, x; u^\varepsilon, v^\varepsilon)$, $j = 1, 2$, are deterministic and $\left\{ (J_1(t, x; u^\varepsilon, v^\varepsilon), J_2(t, x; u^\varepsilon, v^\varepsilon)), \varepsilon > 0 \right\}$ is a bounded sequence. Therefore, we can choose an accumulation point of this sequence, as $\varepsilon \rightarrow 0$, and we denote this point by (e_1, e_2) . Consequently, from Theorem 5.6 it follows that (e_1, e_2) is a Nash equilibrium payoff at (t, x) . The proof is complete. \square

Before proving Proposition 5.10, let us first make some preliminaries for its proof.

Lemma 5.11. *For all $\varepsilon > 0$, $\delta \in [0, T - t]$ and $x \in \mathbb{R}^n$, we have the existence of $(u^\varepsilon, v^\varepsilon) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$ independent of \mathcal{F}_t , such that, $j = 1, 2$,*

$$W_j(t, x) - \varepsilon \leq {}^j G_{t,t+\delta}^{t,x;u^\varepsilon,v^\varepsilon} [W_{N_{t+\delta}^{t,j}}(t + \delta, X_{t+\delta}^{t,x;u^\varepsilon,v^\varepsilon})], \mathbb{P} - a.s.$$

For its proof, we adapt the ideas developed in [16] from SDGs without jumps to SDGs with jumps.

Proof. We denote by $\mathbb{F}^t = (\mathcal{F}_s^t)_{s \in [t, T]}$ the following filtration:

$$\mathcal{F}_s^t := \sigma \left\{ B_r - B_t, N_r - N_t : t \leq r \leq s \right\} \vee \mathcal{N}_{\mathbb{P}}, \quad s \in [t, T],$$

where $\mathcal{N}_{\mathbb{P}}$ is the collection of all \mathbb{P} -null sets. For $s \in [t, T]$, we denote by $\mathcal{U}_{s,T}^t$ (resp., $\mathcal{V}_{s,T}^t$) the set of \mathbb{F}^t -adapted processes $\{u_r\}_{r \in [s, T]}$ (resp., $\{v_r\}_{r \in [s, T]}$) taking their values in U (resp., V), and we let $\mathcal{A}_{s,T}^t$ (resp., $\mathcal{B}_{s,T}^t$) be the NAD strategies from $\mathcal{V}_{s,T}^t$ into $\mathcal{U}_{s,T}^t$ (resp. $\mathcal{U}_{s,T}^t$ into $\mathcal{V}_{s,T}^t$).

Let us replace the framework of SDEs driven by a Brownian motion $B = (B_s)_{s \in [0, T]}$ by that of SDEs driven by a Brownian motion $(B_s - B_t)_{s \in [t, T]}$, and let us also replace the framework of BSDEs driven by a Brownian motion $B = (B_s)_{s \in [0, T]}$ and the Poisson process $(N_s)_{s \in [t, T]}$ by that of $B^t = (B_s - B_t)_{s \in [t, T]}$ and $(N_s - N_t)_{s \in [t, T]}$. Using the arguments of Sections 3 and 4 and Isaacs condition (4.5), we conclude, for $(s, x) \in [t, T] \times \mathbb{R}^n$,

$$\begin{aligned} \widetilde{W}_1(s, x) &= \operatorname{esssup}_{\alpha \in \mathcal{A}_{s,T}^t} \operatorname{essinf}_{\beta \in \mathcal{B}_{s,T}^t} J_1(s, x; \alpha, \beta) = \operatorname{essinf}_{\beta \in \mathcal{B}_{s,T}^t} \operatorname{esssup}_{\alpha \in \mathcal{A}_{s,T}^t} J_1(s, x; \alpha, \beta), \\ \widetilde{W}_2(s, x) &= \operatorname{essinf}_{\alpha \in \mathcal{A}_{s,T}^t} \operatorname{esssup}_{\beta \in \mathcal{B}_{s,T}^t} J_2(s, x; \alpha, \beta) = \operatorname{esssup}_{\beta \in \mathcal{B}_{s,T}^t} \operatorname{essinf}_{\alpha \in \mathcal{A}_{s,T}^t} J_2(s, x; \alpha, \beta). \end{aligned}$$

For $j = 1, 2$, it follows from the Sections 3 and 4 that W_j restricted to $[t, T] \times \mathbb{R}^n$ and \widetilde{W}_j are inside the class of continuous functions with linear growth and the unique viscosity solutions of the same system of Isaacs equations. Therefore,

$$\widetilde{W}_j(s, x) = W_j(s, x), \quad (s, x) \in [t, T] \times \mathbb{R}^n, j = 1, 2.$$

By virtue of the dynamic programming principle for \widetilde{W}_j and $\mathcal{V}_{t,t+\delta}^t \subset \mathcal{B}_{t,t+\delta}^t$ we deduce that

$$\begin{aligned} W_1(t, x) &= \widetilde{W}_1(t, x) = \operatorname{esssup}_{\alpha \in \mathcal{A}_{t,t+\delta}^t} \operatorname{essinf}_{\beta \in \mathcal{B}_{t,t+\delta}^t} {}^1 G_{t,t+\delta}^{t,x;\alpha,\beta} [W_{N_{t+\delta}^{t,1}}(t + \delta, X_{t+\delta}^{t,x;\alpha,\beta})] \\ &\leq \operatorname{esssup}_{\alpha \in \mathcal{A}_{t,t+\delta}^t} \operatorname{essinf}_{v \in \mathcal{V}_{t,t+\delta}^t} {}^1 G_{t,t+\delta}^{t,x;\alpha(v),v} [W_{N_{t+\delta}^{t,1}}(t + \delta, X_{t+\delta}^{t,x;\alpha(v),v})]. \end{aligned}$$

Therefore, for $\varepsilon > 0$ and $\delta > 0$, we have the existence of $\alpha_\varepsilon \in \mathcal{A}_{t,t+\delta}^t$ such that, for all $v \in \mathcal{V}_{t,t+\delta}^t$,

$$W_1(t, x) - \varepsilon \leq {}^1 G_{t,t+\delta}^{t,x;\alpha_\varepsilon(v),v} [W_{N_{t+\delta}^{t,1}}(t + \delta, X_{t+\delta}^{t,x;\alpha_\varepsilon(v),v})], \quad \mathbb{P} - a.s.$$

By a symmetric argument there exists $\beta_\varepsilon \in \mathcal{B}_{t,t+\delta}^t$ such that, for all $u \in \mathcal{U}_{t,t+\delta}^t$,

$$W_2(t, x) - \varepsilon \leq {}^2G_{t,t+\delta}^{t,x;u,\beta_\varepsilon(u)}[W_{N_{t+\delta}^{t,2}}(t + \delta, X_{t+\delta}^{t,x;u,\beta_\varepsilon(u)})], \quad \mathbb{P} - a.s.$$

In analogy to Lemma 3.6, it can be shown that there exists a unique couple $(u^\varepsilon, v^\varepsilon) \in \mathcal{U}_{t,t+\delta}^t \times \mathcal{V}_{t,t+\delta}^t$ such that $\alpha_\varepsilon(v^\varepsilon) = u^\varepsilon, \beta_\varepsilon(u^\varepsilon) = v^\varepsilon$. Consequently,

$$W_j(t, x) - \varepsilon \leq {}^jG_{t,t+\delta}^{t,x;u^\varepsilon,v^\varepsilon}[W_{N_{t+\delta}^{t,j}}(t + \delta, X_{t+\delta}^{t,x;u^\varepsilon,v^\varepsilon})], \quad j = 1, 2.$$

This completes the proof. \square

Also the following Lemma is crucial for the proof of Proposition 5.10.

Lemma 5.12. *For $n \geq 1$, we fix some partition $t = t_0 < t_1 < \dots < t_n = T$ of the interval $[t, T]$. Then, for all $\varepsilon > 0$, there exists $(u^\varepsilon, v^\varepsilon) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$ independent of \mathcal{F}_t , such that, for all $i = 0, \dots, n-1$ and $j = 1, 2$,*

$$W_{N_{t_i}^{t,j}}(t_i, X_{t_i}^{t,x;u^\varepsilon,v^\varepsilon}) - \varepsilon \leq {}^jG_{t_i,t_{i+1}}^{t,x;u^\varepsilon,v^\varepsilon}[W_{N_{t_{i+1}}^{t,j}}(t_{i+1}, X_{t_{i+1}}^{t,x;u^\varepsilon,v^\varepsilon})], \quad \mathbb{P} - a.s.$$

Proof. Let us prove it by induction. Obviously, due to the above lemma, it holds for $i = 0$. Let a couple $(u^\varepsilon, v^\varepsilon)$ independent of \mathcal{F}_t , be constructed on the interval $[t, t_i]$, and we shall give its definition on $[t_i, t_{i+1}]$. By virtue of the above lemma we have, for all $y \in \mathbb{R}^n$, the existence of $(u^y, v^y) \in \mathcal{U}_{t_i,T} \times \mathcal{V}_{t_i,T}$ independent of \mathcal{F}_t , such that,

$$W_j(t_i, y) - \frac{\varepsilon}{2} \leq {}^jG_{t_i,t_{i+1}}^{t_i,y;u^y,v^y}[W_{N_{t_{i+1}}^{t_i,j}}(t_{i+1}, X_{t_{i+1}}^{t_i,y;u^y,v^y})], \quad \mathbb{P} - a.s, j = 1, 2. \quad (5.23)$$

For $j = 1, 2$, $y, z \in \mathbb{R}^n$ and $s \in [t_i, t_{i+1}]$, let us set

$$y_s^1 = {}^jG_{s,t_{i+1}}^{t_i,y;u^y,v^y}[W_{N_{t_{i+1}}^{t_i,j}}(t_{i+1}, X_{t_{i+1}}^{t_i,y;u^y,v^y})], \text{ and } y_s^2 = {}^jG_{s,t_{i+1}}^{t_i,z;u^y,v^y}[W_{N_{t_{i+1}}^{t_i,j}}(t_{i+1}, X_{t_{i+1}}^{t_i,z;u^y,v^y})].$$

Then let us consider the following BSDEs:

$$\begin{aligned} y_s^1 &= W_{N_{t_{i+1}}^{t_i,j}}(t_{i+1}, X_{t_{i+1}}^{t_i,y;u^y,v^y}) + \int_s^{t_{i+1}} f_{N_r^{t_i,j}}(r, X_r^{t_i,y;u^y,v^y}, y_r^1, h_r^1, z_r^1, u_r^y, v_r^y) dr \\ &\quad - \lambda \int_s^{t_{i+1}} h_r^1 dr - \int_s^{t_{i+1}} z_r^1 dB_r - \int_s^{t_{i+1}} h_r^1 d\tilde{N}_r, \quad s \in [t_i, t_{i+1}], \end{aligned}$$

as well as

$$\begin{aligned} y_s^2 &= W_{N_{t_{i+1}}^{t_i,j}}(t_{i+1}, X_{t_{i+1}}^{t_i,z;u^y,v^y}) + \int_s^{t_{i+1}} f_{N_r^{t_i,j}}(r, X_r^{t_i,z;u^y,v^y}, y_r^2, h_r^2, z_r^2, u_r^y, v_r^y) dr \\ &\quad - \lambda \int_s^{t_{i+1}} h_r^2 dr - \int_s^{t_{i+1}} z_r^2 dB_r - \int_s^{t_{i+1}} h_r^2 d\tilde{N}_r, \quad s \in [t_i, t_{i+1}]. \end{aligned}$$

From Lemma 2.3 it follows that

$$\begin{aligned} &|{}^jG_{t_i,t_{i+1}}^{t_i,y;u^y,v^y}[W_{N_{t_{i+1}}^{t_i,j}}(t_{i+1}, X_{t_{i+1}}^{t_i,y;u^y,v^y})] - {}^jG_{t_i,t_{i+1}}^{t_i,z;u^y,v^y}[W_{N_{t_{i+1}}^{t_i,j}}(t_{i+1}, X_{t_{i+1}}^{t_i,z;u^y,v^y})]|^2 \\ &\leq C\mathbb{E}[|W_j(t_{i+1}, X_{t_{i+1}}^{t_i,y;u^y,v^y}) - W_{N_{t_{i+1}}^{t_i,j}}(t_{i+1}, X_{t_{i+1}}^{t_i,z;u^y,v^y})|^2 | \mathcal{F}_{t_i}] \\ &\quad + C\mathbb{E}[|\int_{t_i}^{t_{i+1}} f_{N_r^{t_i,j}}(r, X_r^{t_i,y;u^y,v^y}, y_r^1, h_r^1, z_r^1, u_r^y, v_r^y) dr|] \end{aligned}$$

$$\begin{aligned}
& - \int_{t_i}^{t_{i+1}} f_{N_r^{t_i,j}}(r, X_r^{t_i,z;u^y,v^y}, y_r^1, h_r^1, z_r^1, u_r^y, v_r^y) dr \Big|^2 \Big| \mathcal{F}_{t_i} \Big] \\
& \leq C \mathbb{E} [|X_{t_{i+1}}^{t_i,y;u^y,v^y} - X_{t_{i+1}}^{t_i,z;u^y,v^y}|^2 \Big| \mathcal{F}_t] + C \mathbb{E} \left[\int_{t_i}^{t_{i+1}} |X_r^{t_i,y;u^y,v^y} - X_r^{t_i,z;u^y,v^y}|^2 dr \Big| \mathcal{F}_{t_i} \right] \\
& \leq C |y - z|^2.
\end{aligned}$$

Thus, due to Proposition 3.9 and (5.23) we obtain, for $C|y - z| \leq \frac{\varepsilon}{2}$,

$$\begin{aligned}
W_j(t_i, z) - \varepsilon & \leq W_j(t_i, y) - \varepsilon + C|y - z| \\
& \leq {}^j G_{t_i, t_{i+1}}^{t_i, y; u^y, v^y} [W_{N_{t_{i+1}}^{t_i, j}}(t_{i+1}, X_{t_{i+1}}^{t_i, y; u^y, v^y})] - \frac{\varepsilon}{2} + C|y - z| \\
& \leq {}^j G_{t_i, t_{i+1}}^{t_i, z; u^y, v^y} [W_{N_{t_{i+1}}^{t_i, j}}(t_{i+1}, X_{t_{i+1}}^{t_i, z; u^y, v^y})] - \frac{\varepsilon}{2} + C|y - z| \\
& \leq {}^j G_{t_i, t_{i+1}}^{t_i, z; u^y, v^y} [W_{N_{t_{i+1}}^{t_i, j}}(t_{i+1}, X_{t_{i+1}}^{t_i, z; u^y, v^y})], \quad \mathbb{P} - a.s.
\end{aligned}$$

We let $\{O_i\}_{i \geq 1} \subset \mathcal{B}(\mathbb{R}^n)$ be a partition of \mathbb{R}^n with $\text{diam}(O_i) < \frac{\varepsilon}{2C}$ and let $y_l \in O_l$. Then, for $z \in O_l$, it follows

$$W_j(t_i, z) - \varepsilon \leq {}^j G_{t_i, t_{i+1}}^{t_i, z; u^{y_l}, v^{y_l}} [W_{N_{t_{i+1}}^{t_i, j}}(t_{i+1}, X_{t_{i+1}}^{t_i, z; u^{y_l}, v^{y_l}})], \quad \mathbb{P} - a.s. \quad (5.24)$$

Let us define

$$u^\varepsilon = \sum_{l \geq 1} 1_{O_l}(X_{t_i}^{t, x; u^\varepsilon, v^\varepsilon}) u^{y_l}, \quad v^\varepsilon = \sum_{l \geq 1} 1_{O_l}(X_{t_i}^{t, x; u^\varepsilon, v^\varepsilon}) v^{y_l}.$$

Then, we conclude

$$\begin{aligned}
& {}^j G_{t_i, t_{i+1}}^{t, x; u^\varepsilon, v^\varepsilon} [W_{N_{t_{i+1}}^{t, j}}(t_{i+1}, X_{t_{i+1}}^{t, x; u^\varepsilon, v^\varepsilon})] \\
& = {}^j G_{t_i, t_{i+1}}^{t_i, X_{t_i}^{t, x; u^\varepsilon, v^\varepsilon}; u^\varepsilon, v^\varepsilon} \left[\sum_{l \geq 1} W_{N_{t_{i+1}}^{t_i, j}}(t_{i+1}, X_{t_{i+1}}^{t_i, X_{t_i}^{t, x; u^\varepsilon, v^\varepsilon}; u^\varepsilon, v^\varepsilon}) 1_{O_l}(X_{t_i}^{t, x; u^\varepsilon, v^\varepsilon}) \right] \\
& = {}^j G_{t_i, t_{i+1}}^{t_i, X_{t_i}^{t, x; u^\varepsilon, v^\varepsilon}; u^\varepsilon, v^\varepsilon} \left[\sum_{l \geq 1} W_{N_{t_{i+1}}^{t_i, j}}(t_{i+1}, X_{t_{i+1}}^{t_i, X_{t_i}^{t, x; u^\varepsilon, v^\varepsilon}; u^{y_l}, v^{y_l}) 1_{O_l}(X_{t_i}^{t, x; u^\varepsilon, v^\varepsilon}) \right] \\
& = \sum_{l \geq 1} {}^j G_{t_i, t_{i+1}}^{t_i, X_{t_i}^{t, x; u^\varepsilon, v^\varepsilon}; u^{y_l}, v^{y_l}} [W_{N_{t_{i+1}}^{t_i, j}}(t_{i+1}, X_{t_{i+1}}^{t_i, X_{t_i}^{t, x; u^\varepsilon, v^\varepsilon}; u^{y_l}, v^{y_l})] 1_{O_l}(X_{t_i}^{t, x; u^\varepsilon, v^\varepsilon}).
\end{aligned}$$

From (5.24) it follows that

$$\begin{aligned}
{}^j G_{t_i, t_{i+1}}^{t, x; u^\varepsilon, v^\varepsilon} [W_{N_{t_{i+1}}^{t, j}}(t_{i+1}, X_{t_{i+1}}^{t, x; u^\varepsilon, v^\varepsilon})] & \geq \sum_{l \geq 1} [W_{N_{t_i}^{t, j}}(t_i, X_{t_i}^{t, x; u^\varepsilon, v^\varepsilon}) - \varepsilon] 1_{O_l}(X_{t_i}^{t, x; u^\varepsilon, v^\varepsilon}) \\
& = W_{N_{t_i}^{t, j}}(t_i, X_{t_i}^{t, x; u^\varepsilon, v^\varepsilon}) - \varepsilon,
\end{aligned}$$

from which we conclude the wished result. \square

Let us come, finally, to the proof of Proposition 5.10.

Proof. Let $j = 1, 2$ be arbitrarily fixed, and let τ be a stopping time independent of \mathcal{F}_t , such that $t \leq \tau \leq T$. We put $t_i = \frac{i(T-t)}{2^n} + t$, $A_i = \{t_{i-1} \leq \tau < t_i\}$, $i = 1, \dots, 2^n$, and define

$\tau_n = \sum_{i=1}^{2^n} t_i 1_{A_i}$. It is obvious that $0 \leq \tau_n - \tau \leq 2^{-n}$. From Lemma 5.12 we know that, for all $\varepsilon > 0$, there exists $(u^\varepsilon, v^\varepsilon) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$ independent of \mathcal{F}_t such that, for all stopping time τ independent of \mathcal{F}_t , and for all $i, (0 \leq i \leq 2^n - 1)$,

$$W_{N_{t_i}^{t,j}}(t_i, X_{t_i}^{t,x;u^\varepsilon,v^\varepsilon}) - \varepsilon_0 \leq {}^j G_{t_i, t_{i+1}}^{t,x;u^\varepsilon,v^\varepsilon} [W_{N_{t_{i+1}}^{t,j}}(t_{i+1}, X_{t_{i+1}}^{t,x;u^\varepsilon,v^\varepsilon})], \quad \mathbb{P} - a.s.$$

where $\varepsilon_0 > 0$ depends on ε and n , and will be specified after.

From the Lemmas 2.2 and 2.3 it follows that, for $0 \leq i \leq 2^n - 1$,

$$\begin{aligned} & {}^j G_{t_i, T}^{t,x;u^\varepsilon,v^\varepsilon} [\Phi_{N_T^{t,j}}(X_T^{t,x;u^\varepsilon,v^\varepsilon})] \\ &= {}^j G_{t_i, t_{2^n-1}}^{t,x;u^\varepsilon,v^\varepsilon} [{}^j G_{t_{2^n-1}, T}^{t,x;u^\varepsilon,v^\varepsilon} [\Phi_{N_T^{t,j}}(X_T^{t,x;u^\varepsilon,v^\varepsilon})]] \\ &\geq {}^j G_{t_i, t_{2^n-1}}^{t,x;u^\varepsilon,v^\varepsilon} [W_{N_{t_{2^n-1}}^{t,j}}(t_{2^n-1}, X_{t_{2^n-1}}^{t,x;u^\varepsilon,v^\varepsilon}) - \varepsilon_0] \\ &\geq {}^j G_{t_i, t_{2^n-1}}^{t,x;u^\varepsilon,v^\varepsilon} [W_{N_{t_{2^n-1}}^{t,j}}(t_{2^n-1}, X_{t_{2^n-1}}^{t,x;u^\varepsilon,v^\varepsilon})] - C\varepsilon_0 \\ &\geq \dots \geq {}^j G_{t_i, t_{i+1}}^{t,x;u^\varepsilon,v^\varepsilon} [W_{N_{t_{i+1}}^{t,j}}(t_{i+1}, X_{t_{i+1}}^{t,x;u^\varepsilon,v^\varepsilon})] - C(2^n - i)\varepsilon_0 \\ &\geq W_{N_{t_i}^{t,j}}(t_i, X_{t_i}^{t,x;u^\varepsilon,v^\varepsilon}) - C(2^n - i + 1)\varepsilon_0, \end{aligned}$$

from where

$$\begin{aligned} & {}^j G_{\tau_n, T}^{t,x;u^\varepsilon,v^\varepsilon} [\Phi_{N_T^{t,j}}(X_T^{t,x;u^\varepsilon,v^\varepsilon})] = \sum_{i=1}^{2^n} 1_{A_i} {}^j G_{t_i, T}^{t,x;u^\varepsilon,v^\varepsilon} [\Phi_{N_T^{t,j}}(X_T^{t,x;u^\varepsilon,v^\varepsilon})] \\ &\geq \sum_{i=1}^{2^n} 1_{A_i} W_{N_{t_i}^{t,j}}(t_i, X_{t_i}^{t,x;u^\varepsilon,v^\varepsilon}) - C \sum_{i=1}^{2^n} 1_{A_i} (2^n - i + 1)\varepsilon_0 \\ &\geq \sum_{i=1}^{2^n} 1_{A_i} W_{N_{t_i}^{t,j}}(t_i, X_{t_i}^{t,x;u^\varepsilon,v^\varepsilon}) - C2^n \varepsilon_0 = W_{N_{\tau_n}^{t,j}}(\tau_n, X_{\tau_n}^{t,x;u^\varepsilon,v^\varepsilon}) - C2^n \varepsilon_0. \end{aligned}$$

Therefore,

$$\begin{aligned} {}^j G_{\tau, T}^{t,x;u^\varepsilon,v^\varepsilon} [\Phi_{N_T^{t,j}}(X_T^{t,x;u^\varepsilon,v^\varepsilon})] &= {}^j G_{\tau, \tau_n}^{t,x;u^\varepsilon,v^\varepsilon} [{}^j G_{\tau_n, T}^{t,x;u^\varepsilon,v^\varepsilon} [\Phi_{N_T^{t,j}}(X_T^{t,x;u^\varepsilon,v^\varepsilon})]] \\ &\geq {}^j G_{\tau, \tau_n}^{t,x;u^\varepsilon,v^\varepsilon} [W_{N_{\tau_n}^{t,j}}(\tau_n, X_{\tau_n}^{t,x;u^\varepsilon,v^\varepsilon}) - C2^n \varepsilon_0] \\ &\geq {}^j G_{\tau, \tau_n}^{t,x;u^\varepsilon,v^\varepsilon} [W_{N_{\tau_n}^{t,j}}(\tau_n, X_{\tau_n}^{t,x;u^\varepsilon,v^\varepsilon})] - C2^n \varepsilon_0 \\ &= {}^j G_{\tau, \tau_n}^{t,x;u^\varepsilon,v^\varepsilon} [W_{N_{\tau_n}^{t,j}}(\tau_n, X_{\tau_n}^{t,x;u^\varepsilon,v^\varepsilon})] - \frac{\varepsilon}{2}, \end{aligned} \quad (5.25)$$

for $\varepsilon_0 = \frac{\varepsilon}{C2^{n+1}}$. We denote by

$$\begin{aligned} I_1 &= {}^j G_{\tau, T}^{t,x;u^\varepsilon,v^\varepsilon} [\Phi_{N_T^{t,j}}(X_T^{t,x;u^\varepsilon,v^\varepsilon})] - {}^j G_{\tau, \tau_n}^{t,x;u^\varepsilon,v^\varepsilon} [W_{N_{\tau_n}^{t,j}}(\tau_n, X_{\tau_n}^{t,x;u^\varepsilon,v^\varepsilon})] + \frac{\varepsilon}{2}, \\ I_2 &= {}^j G_{\tau, T}^{t,x;u^\varepsilon,v^\varepsilon} [\Phi_{N_T^{t,j}}(X_T^{t,x;u^\varepsilon,v^\varepsilon})] - W_{N_{\tau}^{t,j}}(\tau, X_{\tau}^{t,x;u^\varepsilon,v^\varepsilon}) + \frac{\varepsilon}{2}, \end{aligned} \quad (5.26)$$

and we put, for $s \in [\tau, \tau_n]$,

$$y_s^1 = {}^j G_{s, \tau_n}^{t,x;u^\varepsilon,v^\varepsilon} [W_{N_{\tau_n}^{t,j}}(\tau_n, X_{\tau_n}^{t,x;u^\varepsilon,v^\varepsilon})].$$

Obviously, $(y_s^1)_{s \in [\tau, \tau_n]}$ is the solution of the following BSDE:

$$y_s^1 = W_{N_{\tau_n}^{t,j}}(\tau_n, X_{\tau_n}^{t,x;u^\varepsilon,v^\varepsilon}) + \int_s^{\tau_n} f_{N_r^{t,j}}(r, X_r^{t,x;u^\varepsilon,v^\varepsilon}, y_r^1, h_r^1, z_r^1, u_r^\varepsilon, v_r^\varepsilon) dr$$

$$-\lambda \int_s^{\tau_n} h_r^1 dr - \int_s^{\tau_n} z_r^1 dB_r - \int_s^{\tau_n} h_r^1 d\tilde{N}_r, \quad s \in [\tau, \tau_n].$$

We will compare y^1 with the process constant in time $y_s^2 = W_{N_\tau^{t,j}}(\tau, X_\tau^{t,x;u^\varepsilon,v^\varepsilon})$, $s \in [\tau, \tau_n]$. We observe that y^2 is the solution of a BSDE which driving coefficient equals to zero. Hence, from Lemma 2.3 it follows that

$$\begin{aligned} & |{}^j G_{\tau,\tau_n}^{t,x;u^\varepsilon,v^\varepsilon}[W_{N_{\tau_n}^{t,j}}(\tau_n, X_{\tau_n}^{t,x;u^\varepsilon,v^\varepsilon})] - W_{N_\tau^{t,j}}(\tau, X_\tau^{t,x;u^\varepsilon,v^\varepsilon})|^2 \\ & \leq C\mathbb{E}[|W_{N_{\tau_n}^{t,j}}(\tau_n, X_{\tau_n}^{t,x;u^\varepsilon,v^\varepsilon}) - W_{N_\tau^{t,j}}(\tau, X_\tau^{t,x;u^\varepsilon,v^\varepsilon})|^2 | \mathcal{F}_\tau] \\ & \quad + C\mathbb{E}[|\int_\tau^{\tau_n} |f_{N_\tau^{t,j}}(r, X_r^{t,x;u^\varepsilon,v^\varepsilon}, y_r^2, 0, 0, u_r^\varepsilon, v_r^\varepsilon)|^2 dr | \mathcal{F}_\tau] \\ & \leq C\mathbb{E}[|W_{N_{\tau_n}^{t,j}}(\tau_n, X_{\tau_n}^{t,x;u^\varepsilon,v^\varepsilon}) - W_{N_\tau^{t,j}}(\tau, X_\tau^{t,x;u^\varepsilon,v^\varepsilon})|^2 | \mathcal{F}_\tau] + C2^{-n}, \end{aligned} \quad (5.27)$$

where we have used the boundedness of $f_i, i = 1, 2$.

By virtue of the Propositions 3.9 and 6.12 we deduce that

$$\begin{aligned} & \mathbb{E}[|W_{N_{\tau_n}^{t,j}}(\tau_n, X_{\tau_n}^{t,x;u^\varepsilon,v^\varepsilon}) - W_{N_\tau^{t,j}}(\tau, X_\tau^{t,x;u^\varepsilon,v^\varepsilon})|^2] \\ & \leq \mathbb{E}[|W_{N_{\tau_n}^{t,j}}(\tau_n, X_{\tau_n}^{t,x;u^\varepsilon,v^\varepsilon}) - W_{N_{\tau_n}^{t,j}}(\tau_n, X_\tau^{t,x;u^\varepsilon,v^\varepsilon})|^2] \\ & \quad + \mathbb{E}[|W_{N_{\tau_n}^{t,j}}(\tau_n, X_\tau^{t,x;u^\varepsilon,v^\varepsilon}) - W_{N_\tau^{t,j}}(\tau, X_\tau^{t,x;u^\varepsilon,v^\varepsilon})|^2] \\ & \leq \mathbb{E}[|X_{\tau_n}^{t,x;u^\varepsilon,v^\varepsilon} - X_\tau^{t,x;u^\varepsilon,v^\varepsilon}|^2] + \mathbb{E}[(1 + |X_\tau^{t,x;u^\varepsilon,v^\varepsilon}|^2)|\tau - \tau_n|] + C\mathbb{E}[1_{\{N_{\tau_n}^{t,j} \neq N_\tau^{t,j}\}}] \\ & \leq C2^{-n}, \end{aligned}$$

where $C > 0$ is a constant which depends on x . Therefore, from (5.27) it follows that

$$\mathbb{E}[|{}^j G_{\tau,\tau_n}^{t,x;u^\varepsilon,v^\varepsilon}[W_{N_{\tau_n}^{t,j}}(\tau_n, X_{\tau_n}^{t,x;u^\varepsilon,v^\varepsilon})] - W_{N_\tau^{t,j}}(\tau, X_\tau^{t,x;u^\varepsilon,v^\varepsilon})|^2] \leq C2^{-n}.$$

Consequently, $\mathbb{E}[|I_1 - I_2|^2] \leq C2^{-n}$. From (5.25) we see that $I_1 \geq 0$. Therefore, by the above inequality we have

$$\mathbb{P}(I_2 \leq -\frac{\varepsilon}{2}) \leq \mathbb{P}(|I_1 - I_2| \geq \frac{\varepsilon}{2}) \leq \frac{4\mathbb{E}[|I_1 - I_2|^2]}{\varepsilon^2} \leq \frac{4C2^{-n}}{\varepsilon^2} \leq \varepsilon,$$

where we choose n such that $4C2^{-n} \leq \varepsilon^3$, i.e., $n \geq (2 + \frac{\ln C - 3 \ln \varepsilon}{\ln 2})$, and from (5.26) we deduce

$$\mathbb{P}\left(W_{N_\tau^{t,j}}(\tau, X_\tau^{t,x;u^\varepsilon,v^\varepsilon}) - \varepsilon \leq {}^j G_{\tau,T}^{t,x;u^\varepsilon,v^\varepsilon}[\Phi_{N_T^{t,j}}(X_T^{t,x;u^\varepsilon,v^\varepsilon})]\right) > 1 - \varepsilon, \text{ i.e.,}$$

$$\mathbb{P}\left(W_{N_\tau^{t,j}}(\tau, X_\tau^{t,x;u^\varepsilon,v^\varepsilon}) - \varepsilon \leq {}^j Y_\tau^{t,x;u^\varepsilon,v^\varepsilon}\right) > 1 - \varepsilon.$$

Since $(u^\varepsilon, v^\varepsilon)$ as well as τ are independent of \mathcal{F}_t , the event $\{W_{N_\tau^{t,j}}(\tau, X_\tau^{t,x;u^\varepsilon,v^\varepsilon}) - \varepsilon \leq {}^j Y_\tau^{t,x;u^\varepsilon,v^\varepsilon}\}$ is independent of \mathcal{F}_t , from which we see that the conditional probability $\mathbb{P}(\cdot | \mathcal{F}_t)$ of $\{W_{N_\tau^{t,j}}(\tau, X_\tau^{t,x;u^\varepsilon,v^\varepsilon}) - \varepsilon \leq {}^j Y_\tau^{t,x;u^\varepsilon,v^\varepsilon}\}$ coincides with its probability. Consequently,

$$\mathbb{P}\left(W_{N_\tau^{t,j}}(\tau, X_\tau^{t,x;u^\varepsilon,v^\varepsilon}) - \varepsilon \leq {}^j Y_\tau^{t,x;u^\varepsilon,v^\varepsilon} | \mathcal{F}_t\right) > 1 - \varepsilon.$$

The proof is complete. \square

6 Proof of the Theorems 3.10 and 4.2

6.1 Proof of Theorem 3.10

We have postponed to this section the proof of the DPP. We shall establish it first for deterministic times, then we deduce the general version for stopping times.

6.1.1 Dynamic programming principle for deterministic times

Theorem 6.1. *Under the assumptions (H4) and (H5), the following dynamic programming principle (DPP) holds: For any $0 \leq t < t + \delta \leq T$, $x \in \mathbb{R}^n$, $i = 1, 2$,*

$$\begin{aligned} W_i(t, x) &= \operatorname{esssup}_{\alpha \in \mathcal{A}_{t, t+\delta}} \operatorname{essinf}_{\beta \in \mathcal{B}_{t, t+\delta}} {}^i G_{t, t+\delta}^{t, x; \alpha, \beta} [W_{N_{t+\delta}^{t, i}}(t + \delta, X_{t+\delta}^{t, x; \alpha, \beta})]. \\ U_i(t, x) &= \operatorname{essinf}_{\beta \in \mathcal{B}_{t, t+\delta}} \operatorname{esssup}_{\alpha \in \mathcal{A}_{t, t+\delta}} {}^i G_{t, t+\delta}^{t, x; \alpha, \beta} [U_{N_{t+\delta}^{t, i}}(t + \delta, X_{t+\delta}^{t, x; \alpha, \beta})]. \end{aligned} \quad (6.1)$$

Remark 6.2. Recall that for $i = 1, 2$, $(t, x) \in [0, T] \times \mathbb{R}^n$, $W(t, (x, i)) := W_i(t, x)$ is the value function of the stochastic differential game which dynamics is given by the process $(X^{t, x; u, v}, N^{t, i})$, and which cost functional is defined by our BSDE (3.4). The DPP for games with jumps but of the type "strategy against control" was proved in [9], and before, in another framework by Biswas [5]. However, unlike [9] we have to deal here with games of the type "NAD strategy against NAD strategy".

Proof. We only give for arbitrarily fixed $i = 1, 2$, the proof for $W_i(t, x)$, since for $U_i(t, x)$ we can use a symmetric argument. Let us denote by $W_\delta(t, x)$ the right hand side of (6.1). Using the arguments in Proposition 3.7 we can show that $W_\delta(t, x)$ is deterministic. We split now the proof into the following two lemmas. \square

Lemma 6.3. $W_i(t, x) \leq W_\delta(t, x)$.

Proof. From the definition of $W_\delta(t, x)$ it follows that

$$\begin{aligned} W_\delta(t, x) &= \operatorname{esssup}_{\alpha_1 \in \mathcal{A}_{t, t+\delta}} \operatorname{essinf}_{\beta_1 \in \mathcal{B}_{t, t+\delta}} {}^i G_{t, t+\delta}^{t, x; \alpha_1, \beta_1} [W_{N_{t+\delta}^{t, i}}(t + \delta, X_{t+\delta}^{t, x; \alpha_1, \beta_1})] \\ &= \operatorname{esssup}_{\alpha_1 \in \mathcal{A}_{t, t+\delta}} I_\delta(t, x, \alpha_1), \end{aligned}$$

where $I_\delta(t, x, \alpha_1) = \operatorname{essinf}_{\beta_1 \in \mathcal{B}_{t, t+\delta}} {}^i G_{t, t+\delta}^{t, x; \alpha_1, \beta_1} [W_{N_{t+\delta}^{t, i}}(t + \delta, X_{t+\delta}^{t, x; \alpha_1, \beta_1})]$. Then, there exists a sequence $\{\alpha_n^1, n \geq 1\} \subset \mathcal{A}_{t, t+\delta}$, such that $W_\delta(t, x) = \sup_{n \geq 1} I_\delta(t, x, \alpha_n^1)$, \mathbb{P} -a.s. For any $\varepsilon > 0$, let us put

$$\Lambda_n := \left\{ I_\delta(t, x, \alpha_n^1) + \varepsilon \geq W_\delta(t, x), I_\delta(t, x, \alpha_j^1) + \varepsilon < W_\delta(t, x), 1 \leq j \leq n-1 \right\} \in \mathcal{F}_t, \quad n \geq 1.$$

Obviously, $\{\Lambda_n, n \geq 1\}$ is an (Ω, \mathcal{F}_t) -partition. We define $\alpha_1^\varepsilon := \sum_{n \geq 1} 1_{\Lambda_n} \alpha_n^1$. By a straightforward proof it can be shown that α_1^ε belongs to $\mathcal{A}_{t, t+\delta}$. The uniqueness of the FBSDE and the definition of NAD strategies allows to show that, for all $\beta_1 \in \mathcal{B}_{t, t+\delta}$, $X^{t, x; \alpha_1^\varepsilon, \beta_1} = \sum_{n \geq 1} 1_{\Lambda_n} X^{t, x; \alpha_n^1, \beta_1}$. Indeed, let $(u^n, v^n) \in \mathcal{U}_{t, T} \times \mathcal{V}_{t, T}$ be associated with (α_n^1, β_1) by Lemma 3.6, and put

$$u_1^\varepsilon := \sum_{n \geq 1} 1_{\Lambda_n} u^n, \quad v_1^\varepsilon := \sum_{n \geq 1} 1_{\Lambda_n} v^n.$$

Then straight forward arguments allow to verify that $(u_1^\varepsilon, v_1^\varepsilon) \in \mathcal{U}_{t, T} \times \mathcal{V}_{t, T}$ is associated with $(\alpha^\varepsilon, \beta_1)$ by Lemma 3.6 such that $\alpha^\varepsilon(v_1^\varepsilon) = u_1^\varepsilon$, $\beta_1(u_1^\varepsilon) = v_1^\varepsilon$. Consequently,

$${}^i G_{t, t+\delta}^{t, x; \alpha_1^\varepsilon, \beta_1} [W_{N_{t+\delta}^{t, i}}(t + \delta, X_{t+\delta}^{t, x; \alpha_1^\varepsilon, \beta_1})] = \sum_{n \geq 1} {}^i G_{t, t+\delta}^{t, x; \alpha_n^1, \beta_1} [W_{N_{t+\delta}^{t, i}}(t + \delta, X_{t+\delta}^{t, x; \alpha_n^1, \beta_1})] 1_{\Lambda_n}. \quad (6.2)$$

Therefore, for all $\beta_1 \in \mathcal{B}_{t,t+\delta}$,

$$\begin{aligned}
W_\delta(t, x) &\leq \sum_{n \geq 1} 1_{\Lambda_n} I_\delta(t, x, \alpha_n^1) + \varepsilon \\
&\leq \sum_{n \geq 1} 1_{\Lambda_n} {}^i G_{t,t+\delta}^{t,x;\alpha_n^1,\beta_1} [W_{N_{t+\delta}^{t,i}}(t + \delta, X_{t+\delta}^{t,x;\alpha_n^1,\beta_1})] + \varepsilon \\
&= {}^i G_{t,t+\delta}^{t,x;\alpha_1^\varepsilon,\beta_1} [W_{N_{t+\delta}^{t,i}}(t + \delta, X_{t+\delta}^{t,x;\alpha_1^\varepsilon,\beta_1})] + \varepsilon, \quad \mathbb{P}\text{-a.s.}
\end{aligned} \tag{6.3}$$

We let $\{O_j\}_{j \geq 1} \subset \mathcal{B}(\mathbb{R}^n)$ be a partition of \mathbb{R}^n such that $\sum_{j \geq 1} O_j = \mathbb{R}^n$ and $\text{diam}(O_j) \leq \varepsilon$, $j \geq 1$.

Let us fix an element y_j of O_j , $j \geq 1$. Then, from the definition of $W_k(t + \delta, y)$, $k = 1, 2$, it follows (through a procedure already used above) that there exists $\alpha_2^{j,k} \in \mathcal{A}_{t+\delta,T}$ such that, for all $\beta_2 \in \mathcal{B}_{t+\delta,T}$,

$$W_k(t + \delta, y_j) \leq \text{essinf}_{\beta_2 \in \mathcal{B}_{t+\delta,T}} J_k(t + \delta, y_j; \alpha_2^{j,k}, \beta_2) + \varepsilon, \quad \mathbb{P}\text{-a.s.}$$

Therefore, for all $\beta_2 \in \mathcal{B}_{t+\delta,T}$,

$$\begin{aligned}
W_{N_{t+\delta}^{t,i}}(t + \delta, y_j) &= \sum_{k=1}^2 1_{\{N_{t+\delta}^{t,i}=k\}} W_k^{t,i}(t + \delta, y_j) \\
&\leq \sum_{k=1}^2 1_{\{N_{t+\delta}^{t,i}=k\}} J_k(t + \delta, y_j; \alpha_2^{j,k}, \beta_2) + \varepsilon \\
&\leq J_{N_{t+\delta}^{t,i}}(t + \delta, y_j; \alpha_2^{j,N_{t+\delta}^{t,i}}, \beta_2) + \varepsilon, \quad \mathbb{P}\text{-a.s.}
\end{aligned} \tag{6.4}$$

For $\beta \in \mathcal{B}_{t,T}$ and $u_2 \in \mathcal{U}_{t+\delta,T}$, we let $\beta_1(u_1) := \beta(u_1 \oplus u_2)|_{[t,t+\delta]}$, $u_1 \in \mathcal{U}_{t,t+\delta}$, where $u_1 \oplus u_2 = u_1 1_{[t,t+\delta]} + u_2 1_{(t+\delta,T]}$. Then $\beta_1 \in \mathcal{B}_{t,t+\delta}$. Notice that β_1 doesn't depend on u_2 thanks to the nonanticipativity of β . Since $(\alpha_1^\varepsilon, \beta_1) \in \mathcal{A}_{t,t+\delta} \times \mathcal{B}_{t,t+\delta}$, we know from Lemma 3.6 that there exists a unique pair $(u_1^\varepsilon, v_1^\varepsilon) \in \mathcal{U}_{t,t+\delta} \times \mathcal{V}_{t,t+\delta}$ such that $\alpha_1^\varepsilon(v_1^\varepsilon) = u_1^\varepsilon$ and $\beta_1(u_1^\varepsilon) = v_1^\varepsilon$. Let us define

$$\beta_2^\varepsilon(u_2) := \beta(u_1^\varepsilon \oplus u_2)|_{(t+\delta,T]}, \quad \text{for } u_2 \in \mathcal{U}_{t+\delta,T}, \quad \alpha_2^\varepsilon := \sum_{j \geq 1} 1_{O_j} (X_{t+\delta}^{t,x;\alpha_1^\varepsilon,\beta_1}) \alpha_2^{j,N_{t+\delta}^{t,i}}.$$

Moreover, we put $\alpha^\varepsilon(v) := \alpha_1^\varepsilon(v_1) \oplus \alpha_2^\varepsilon(v_2)$, for $v = v_1 \oplus v_2$, $v_1 \in \mathcal{V}_{t,t+\delta}$, $v_2 \in \mathcal{V}_{t+\delta,T}$. Since $\alpha_1^\varepsilon \in \mathcal{A}_{t,t+\delta}$ and $\alpha_2^\varepsilon \in \mathcal{A}_{t+\delta,T}$ are NAD strategies, α^ε is also an NAD strategy.

From Proposition 3.9 it follows that

$$W_{N_{t+\delta}^{t,i}}(t + \delta, X_{t+\delta}^{t,x;\alpha_1^\varepsilon,\beta_1}) \leq \sum_{j \geq 1} 1_{O_j} (X_{t+\delta}^{t,x;\alpha_1^\varepsilon,\beta_1}) W_{N_{t+\delta}^{t,i}}(t + \delta, y_j) + C\varepsilon,$$

which together with (6.3), (6.4) and Lemma 2.2 yields

$$\begin{aligned}
W_\delta(t, x) &\leq {}^i G_{t,t+\delta}^{t,x;\alpha_1^\varepsilon,\beta_1} [W_{N_{t+\delta}^{t,i}}(t + \delta, X_{t+\delta}^{t,x;\alpha_1^\varepsilon,\beta_1})] + \varepsilon, \\
&\leq {}^i G_{t,t+\delta}^{t,x;\alpha_1^\varepsilon,\beta_1} [\sum_{j \geq 1} 1_{O_j} (X_{t+\delta}^{t,x;\alpha_1^\varepsilon,\beta_1}) W_{N_{t+\delta}^{t,i}}(t + \delta, y_j) + C\varepsilon] + \varepsilon, \quad \mathbb{P}\text{-a.s.}
\end{aligned}$$

It follows from (6.4) and the Lemmas 2.2, 2.3 and 3.6 that

$$W_\delta(t, x) \leq {}^i G_{t,t+\delta}^{t,x;\alpha_1^\varepsilon,\beta_1} [\sum_{j \geq 1} 1_{O_j} (X_{t+\delta}^{t,x;\alpha_1^\varepsilon,\beta_1}) W_{N_{t+\delta}^{t,i}}(t + \delta, y_j)] + C\varepsilon$$

$$\begin{aligned}
&\leq {}^i G_{t,t+\delta}^{t,x;\alpha_1^\varepsilon,\beta_1} \left[\sum_{j \geq 1} 1_{O_j}(X_{t+\delta}^{t,x;\alpha_1^\varepsilon,\beta_1}) J_{N_{t+\delta}^{t,i}}(t+\delta, y_j; \alpha_2^{j,N_{t+\delta}^{t,i}}, \beta_2) + \varepsilon \right] + C\varepsilon \\
&\leq {}^i G_{t,t+\delta}^{t,x;\alpha_1^\varepsilon,\beta_1} \left[\sum_{j \geq 1} 1_{O_j}(X_{t+\delta}^{t,x;\alpha_1^\varepsilon,\beta_1}) J_{N_{t+\delta}^{t,i}}(t+\delta, y_j; \alpha_2^\varepsilon, \beta_2) \right] + C\varepsilon \\
&\leq {}^i G_{t,t+\delta}^{t,x;\alpha_1^\varepsilon,\beta_1} [J_{N_{t+\delta}^{t,i}}(t+\delta, X_{t+\delta}^{t,x;\alpha_1^\varepsilon,\beta_1}; \alpha_2^\varepsilon, \beta_2) + C\varepsilon] + C\varepsilon \\
&\leq {}^i G_{t,t+\delta}^{t,x;\alpha_1^\varepsilon,\beta_1} [J_{N_{t+\delta}^{t,i}}(t+\delta, X_{t+\delta}^{t,x;\alpha_1^\varepsilon,\beta_1}; \alpha_2^\varepsilon, \beta_2)] + C\varepsilon \\
&= {}^i G_{t,t+\delta}^{t,x;\alpha_1^\varepsilon,\beta_1} [N_{t+\delta}^{t,i} Y_{t+\delta}^{t+\delta, X_{t+\delta}^{t,x;\alpha_1^\varepsilon,\beta_1}; \alpha_2^\varepsilon, \beta_2}] + C\varepsilon \\
&= {}^i G_{t,t+\delta}^{t,x;\alpha_1^\varepsilon,\beta_1} [{}^i Y_{t+\delta}^{t,x;\alpha_2^\varepsilon, \beta_2}] + C\varepsilon \\
&= J_i(t, x; \alpha^\varepsilon, \beta) + C\varepsilon, \quad \mathbb{P}\text{-a.s. for all } \beta \in \mathcal{B}_{t,T}.
\end{aligned}$$

Consequently, due to the choice of α^ε we conclude $W_\delta(t, x) \leq W_i(t, x) + C\varepsilon$, \mathbb{P} -a.s. Letting $\varepsilon \downarrow 0$ we have $W_\delta(t, x) \leq W_i(t, x)$. \square

In order to complete the proof of Theorem 6.1 we have to prove the converse inequality.

Lemma 6.4. $W_\delta(t, x) \geq W_i(t, x)$.

Proof. For an arbitrarily given $\alpha \in \mathcal{A}_{t,T}$ and a given $v_2(\cdot) \in \mathcal{V}_{t+\delta,T}$, let us define

$$\alpha_1(v_1) := \alpha(v_1 \oplus v_2)|_{[t,t+\delta]}, \quad v_1(\cdot) \in \mathcal{V}_{t,t+\delta},$$

where $v_1 \oplus v_2 := v_1 1_{[t,t+\delta]} + v_2 1_{(t+\delta,T]}$. Then $\alpha_1 \in \mathcal{A}_{t,t+\delta}$, and α_1 does not depend on the choice of $v_2(\cdot) \in \mathcal{V}_{t+\delta,T}$, since α is nonanticipative. Therefore, by virtue of the definition of $W_\delta(t, x)$ we know that

$$W_\delta(t, x) \geq \text{essinf}_{\beta_1 \in \mathcal{B}_{t,t+\delta}} {}^i G_{t,t+\delta}^{t,x;\alpha_1,\beta_1} [W_{N_{t+\delta}^{t,i}}(t+\delta, X_{t+\delta}^{t,x;\alpha_1,\beta_1})], \quad \mathbb{P}\text{-a.s.}$$

Let us denote by $I_\delta(t, x, \alpha_1, \beta_1) := {}^i G_{t,t+\delta}^{t,x;\alpha_1,\beta_1} [W_{N_{t+\delta}^{t,i}}(t+\delta, X_{t+\delta}^{t,x;\alpha_1,\beta_1})]$. Then there exists a sequence $\{\beta_n^1, n \geq 1\} \subset \mathcal{B}_{t,t+\delta}$ such that

$$I_\delta(t, x, \alpha_1) := \text{essinf}_{\beta_1 \in \mathcal{B}_{t,t+\delta}} I_\delta(t, x, \alpha_1, \beta_1) = \inf_{n \geq 1} I_\delta(t, x, \alpha_1, \beta_n^1), \quad \mathbb{P}\text{-a.s.}$$

For any $\varepsilon > 0$, $n \geq 1$, we let

$$\Lambda_n := \{I_\delta(t, x, \alpha_1) \geq I_\delta(t, x, \alpha_1, \beta_n^1) - \varepsilon, I_\delta(t, x, \alpha_1) < I_\delta(t, x, \alpha_1, \beta_j^1) - \varepsilon, 1 \leq j \leq n-1\} \in \mathcal{F}_t.$$

Then, $\{\Lambda_n\}$ is a partition of (Ω, \mathcal{F}_t) . We define $\beta_1^\varepsilon := \sum_{n \geq 1} 1_{\Lambda_n} \beta_n^1$, and it can be proven that $\beta_1^\varepsilon \in \mathcal{B}_{t,t+\delta}$. Thanks to the uniqueness of the solution of the FBSDE we have

$$I_\delta(t, x, \alpha_1, \beta_1^\varepsilon) = \sum_{n \geq 1} 1_{\Lambda_n} I_\delta(t, x, \alpha_1, \beta_n^1), \quad \mathbb{P}\text{-a.s.}$$

Indeed, this relation can be proved with an argument analogous to that for (6.2). Therefore,

$$\begin{aligned}
W_\delta(t, x) \geq I_\delta(t, x, \alpha_1) &\geq \sum_{n \geq 1} 1_{\Lambda_n} I_\delta(t, x, \alpha_1, \beta_n^1) - \varepsilon = I_\delta(t, x, \alpha_1, \beta_1^\varepsilon) - \varepsilon \\
&= {}^i G_{t,t+\delta}^{t,x;\alpha_1,\beta_1^\varepsilon} [W_{N_{t+\delta}^{t,i}}(t+\delta, X_{t+\delta}^{t,x;\alpha_1,\beta_1^\varepsilon})] - \varepsilon, \quad \mathbb{P}\text{-a.s.}
\end{aligned}$$

Since $(\alpha_1, \beta_1^\varepsilon) \in \mathcal{A}_{t,t+\delta} \times \mathcal{B}_{t,t+\delta}$, by Lemma 3.6 there exists a unique pair $(u_1^\varepsilon, v_1^\varepsilon) \in \mathcal{U}_{t,t+\delta} \times \mathcal{V}_{t,t+\delta}$ such that $\alpha_1(v_1^\varepsilon) = u_1^\varepsilon$ and $\beta_1^\varepsilon(u_1^\varepsilon) = v_1^\varepsilon$. We also define $\alpha_2^\varepsilon(v_2) := \alpha(v_1^\varepsilon \oplus v_2)|_{[t+\delta,T]}$, $v_2 \in \mathcal{V}_{t+\delta,T}$.

For any $y \in \mathbb{R}^n, k = 1, 2$, from the definition of $W_k(t + \delta, y)$ it follows that

$$W_k(t + \delta, y) \geq \text{essinf}_{\beta_2 \in \mathcal{B}_{t+\delta, T}} J_k(t + \delta, y; \alpha_2^\varepsilon, \beta_2), \quad \mathbb{P}\text{-a.s.}, \quad k = 1, 2.$$

Using the Lipschitz continuity of $W_k(t + \delta, \cdot)$ and $J_k(t + \delta, \cdot)$ we can prove by approximating $X_{t+\delta}^{t, x; \alpha_1, \beta_1^\varepsilon}$ by a finite-valued random variable that

$$W_k(t + \delta, X_{t+\delta}^{t, x; \alpha_1, \beta_1^\varepsilon}) \geq \text{essinf}_{\beta_2 \in \mathcal{B}_{t+\delta, T}} J_k(t + \delta, X_{t+\delta}^{t, x; \alpha_1, \beta_1^\varepsilon}; \alpha_2^\varepsilon, \beta_2), \quad \mathbb{P}\text{-a.s.}, \quad k = 1, 2.$$

Therefore,

$$W_{N_{t+\delta}^{t, i}}(t + \delta, X_{t+\delta}^{t, x; \alpha_1, \beta_1^\varepsilon}) \geq \text{essinf}_{\beta_2 \in \mathcal{B}_{t+\delta, T}} J_{N_{t+\delta}^{t, i}}(t + \delta, X_{t+\delta}^{t, x; \alpha_1, \beta_1^\varepsilon}; \alpha_2^\varepsilon, \beta_2), \quad \mathbb{P}\text{-a.s.} \quad (6.5)$$

Moreover, there exists some sequence $\{\beta_n^2, n \geq 1\} \subset \mathcal{B}_{t+\delta, T}$ such that

$$\text{essinf}_{\beta_2 \in \mathcal{B}_{t+\delta, T}} J_{N_{t+\delta}^{t, i}}(t + \delta, X_{t+\delta}^{t, x; \alpha_1, \beta_1^\varepsilon}; \alpha_2^\varepsilon, \beta_2) = \inf_{n \geq 1} J_{N_{t+\delta}^{t, i}}(t + \delta, X_{t+\delta}^{t, x; \alpha_1, \beta_1^\varepsilon}; \alpha_2^\varepsilon, \beta_n^2), \quad \mathbb{P}\text{-a.s.},$$

and we set, for $n \geq 1$,

$$\begin{aligned} \Delta_n : &= \left\{ \text{essinf}_{\beta_2 \in \mathcal{B}_{t+\delta, T}} J_{N_{t+\delta}^{t, i}}(t + \delta, X_{t+\delta}^{t, x; \alpha_1, \beta_1^\varepsilon}; \alpha_2^\varepsilon, \beta_2) \geq J_{N_{t+\delta}^{t, i}}(t + \delta, X_{t+\delta}^{t, x; \alpha_1, \beta_1^\varepsilon}; \alpha_2^\varepsilon, \beta_n^2) - \varepsilon, \right. \\ &\quad \left. \text{essinf}_{\beta_2 \in \mathcal{B}_{t+\delta, T}} J_{N_{t+\delta}^{t, i}}(t + \delta, X_{t+\delta}^{t, x; \alpha_1, \beta_1^\varepsilon}; \alpha_2^\varepsilon, \beta_2) < J_{N_{t+\delta}^{t, i}}(t + \delta, X_{t+\delta}^{t, x; \alpha_1, \beta_1^\varepsilon}; \alpha_2^\varepsilon, \beta_j^2) - \varepsilon, \right. \\ &\quad \left. 1 \leq j \leq n - 1 \right\} \in \mathcal{F}_{t+\delta}. \end{aligned}$$

Obviously, $\{\Delta_n, n \geq 1\}$ is a partition of $(\Omega, \mathcal{F}_{t+\delta})$. Let us define

$$\beta_2^\varepsilon := \sum_{n \geq 1} 1_{\Delta_n} \beta_n^2, \quad \beta^\varepsilon(u_1 \oplus u_2) := \beta_1^\varepsilon(u_1) \oplus \beta_2^\varepsilon(u_2),$$

for $u_1 \in \mathcal{U}_{t, t+\delta}$ and $u_2 \in \mathcal{U}_{t+\delta, T}$. Then, from the uniqueness for the equations (3.1) and (3.4), combined with Lemma 3.6, it follows that

$$\begin{aligned} J_{N_{t+\delta}^{t, i}}(t + \delta, X_{t+\delta}^{t, x; \alpha_1, \beta_1^\varepsilon}; \alpha_2^\varepsilon, \beta_2^\varepsilon) &= N_{t+\delta}^{t, i} Y_{t+\delta}^{t+\delta, X_{t+\delta}^{t, x; \alpha_1, \beta_1^\varepsilon}; \alpha_2^\varepsilon, \beta_2^\varepsilon} \\ &= \sum_{n \geq 1} 1_{\Delta_n} J_{N_{t+\delta}^{t, i}}(t + \delta, X_{t+\delta}^{t, x; \alpha_1, \beta_1^\varepsilon}; \alpha_2^\varepsilon, \beta_n^2), \quad \mathbb{P}\text{-a.s.}, \end{aligned}$$

which together with (6.5) yields

$$\begin{aligned} W_{N_{t+\delta}^{t, i}}(t + \delta, X_{t+\delta}^{t, x; \alpha_1, \beta_1^\varepsilon}) &\geq \text{essinf}_{\beta_2 \in \mathcal{B}_{t+\delta, T}} J_{N_{t+\delta}^{t, i}}(t + \delta, X_{t+\delta}^{t, x; \alpha_1, \beta_1^\varepsilon}; \alpha_2^\varepsilon, \beta_2) \\ &\geq \sum_{n \geq 1} 1_{\Delta_n} J_{N_{t+\delta}^{t, i}}(t + \delta, X_{t+\delta}^{t, x; \alpha_1, \beta_1^\varepsilon}; \alpha_2^\varepsilon, \beta_n^2) - \varepsilon \\ &= J_{N_{t+\delta}^{t, i}}(t + \delta, X_{t+\delta}^{t, x; \alpha_1, \beta_1^\varepsilon}; \alpha_2^\varepsilon, \beta_2^\varepsilon) - \varepsilon \\ &= N_{t+\delta}^{t, i} Y_{t+\delta}^{t+\delta, X_{t+\delta}^{t, x; \alpha_1, \beta_1^\varepsilon}; \alpha_2^\varepsilon, \beta_2^\varepsilon} - \varepsilon. \end{aligned}$$

Let $\alpha^\varepsilon(v_1 \oplus v_2) := \alpha_1^\varepsilon(v_1) \oplus \alpha_2^\varepsilon(v_2)$, $v_1 \in \mathcal{V}_{t, t+\delta}, v_2 \in \mathcal{V}_{t+\delta, T}$. Then $\alpha^\varepsilon(v)|_{[t, t+\delta]} = \alpha(v)|_{[t, t+\delta]}$, $v \in \mathcal{V}_{t, T}$. Thus,

$$N_{t+\delta}^{t, i} Y_{t+\delta}^{t+\delta, X_{t+\delta}^{t, x; \alpha_1, \beta_1^\varepsilon}; \alpha_2^\varepsilon, \beta_2^\varepsilon} = i Y_{t+\delta}^{t, x; \alpha^\varepsilon, \beta^\varepsilon}.$$

Consequently, by virtue of the Lemmas 2.2 and 2.3 we conclude

$$\begin{aligned}
W_\delta(t, x) \geq I_\delta(t, x, \alpha_1) &\geq {}^iG_{t, t+\delta}^{t, x; \alpha_1, \beta_1^\varepsilon} [W_{N_{t+\delta}^{t, i}}(t + \delta, X_{t+\delta}^{t, x; \alpha_1, \beta_1^\varepsilon})] - \varepsilon \\
&\geq {}^iG_{t, t+\delta}^{t, x; \alpha_1, \beta_1^\varepsilon} [{}^iY_{t+\delta}^{t, x; \alpha^\varepsilon, \beta^\varepsilon} - \varepsilon] - \varepsilon \\
&\geq {}^iG_{t, t+\delta}^{t, x; \alpha^\varepsilon, \beta^\varepsilon} [{}^iY_{t+\delta}^{t, x; \alpha^\varepsilon, \beta^\varepsilon}] - C\varepsilon \\
&= {}^iY_t^{t, x; \alpha^\varepsilon, \beta^\varepsilon} - C\varepsilon \\
&= {}^iY_t^{t, x; \alpha, \beta^\varepsilon} - C\varepsilon,
\end{aligned}$$

where, for the latter equality, we have used Lemma 3.6. Indeed, letting $(u_2^\varepsilon, v_2^\varepsilon) \in \mathcal{U}_{t+\delta, T} \times \mathcal{V}_{t+\delta, T}$ be associated with $(\alpha_2^\varepsilon, \beta_2^\varepsilon)$ by Lemma 3.6 we have

$$\begin{aligned}
\beta^\varepsilon(u_1^\varepsilon \oplus u_2^\varepsilon) &= \beta_1^\varepsilon(u_1^\varepsilon) \oplus \beta_2^\varepsilon(u_2^\varepsilon) = v_1^\varepsilon \oplus v_2^\varepsilon, \\
\alpha^\varepsilon(v_1^\varepsilon \oplus v_2^\varepsilon) &= \alpha_1^\varepsilon(v_1^\varepsilon) \oplus \alpha^\varepsilon(v_1^\varepsilon \oplus v_2^\varepsilon)|_{[t+\delta, T]} = u_1^\varepsilon \oplus u_2^\varepsilon,
\end{aligned}$$

but also $\alpha^\varepsilon(v_1^\varepsilon \oplus v_2^\varepsilon) = \alpha(v_1^\varepsilon \oplus v_2^\varepsilon)$. Recalling now the arbitrariness of $\alpha \in \mathcal{A}_{t, T}$ we conclude that $W_\delta(t, x) \geq W_i(t, x) - C\varepsilon$. Finally, letting $\varepsilon \downarrow 0$ we have $W_\delta(t, x) \geq W_i(t, x)$. \square

From the dynamic programming principle (Theorem 6.1) and standard arguments for BSDEs (see Peng [21] or Buckdahn and Li [10]) we obtain the following proposition.

Proposition 6.5. *Under the assumptions (H4) and (H5), $W(t, x)$ is $\frac{1}{2}$ -Hölder continuous in t , i.e., there exists a positive constant C such that, for all $x \in \mathbb{R}^n$, $t, t' \in [0, T]$,*

$$|W_i(t, x) - W_i(t', x)| \leq C(1 + |x|)|t - t'|^{\frac{1}{2}}.$$

6.1.2 Dynamic programming principle for stopping times

The objective of this subsection is to obtain the dynamic programming principle for stopping times. For this end, we need the following proposition, which turns out to be crucial in our approach.

Proposition 6.6. *For $0 \leq t \leq T$, let τ be a stopping time such that $t \leq \tau \leq T$. Then we have*

$$\text{esssup}_{\alpha_1 \in \mathcal{A}_{\tau, T}} \text{essinf}_{\beta_1 \in \mathcal{B}_{\tau, T}} {}^iY_\tau^{\tau, x, \alpha_1, \beta_1} = \text{esssup}_{\alpha \in \mathcal{A}_{t, T}} \text{essinf}_{\beta \in \mathcal{B}_{t, T}} {}^iY_\tau^{\tau, x, \alpha, \beta}.$$

Proof. We give the proof in two steps.

Step 1: For all $\alpha \in \mathcal{A}_{t, T}$ and an arbitrary fixed $v^0 \in \mathcal{V}_{t, \tau}$, let us define

$$\alpha_1(v^1) := \alpha(v^0 \oplus v^1)|_{[\tau, T]}, v^1 \in \mathcal{V}_{\tau, T}, \text{ and } \alpha_0(v^0) := \alpha(v^0 \oplus v^1)|_{[t, \tau]}, v^1 \in \mathcal{V}_{\tau, T}.$$

It is straight-forward to check that $\alpha_1 \in \mathcal{A}_{\tau, T}$. Since α is nonanticipative, $\alpha_0(v^0)$ only depends on v^0 , but not on v^1 . We put $u^0 := \alpha_0(v^0)$ and define, for $u \in \mathcal{U}_{t, T}$ and $\beta_1 \in \mathcal{B}_{\tau, T}$ $\beta(u) := v^0 \oplus \beta_1(u^1)$, $u^1 = u|_{[\tau, T]}$. Then $\beta \in \mathcal{B}_{t, T}$.

Since $\alpha_1 \in \mathcal{A}_{\tau, T}$ and $\beta_1 \in \mathcal{B}_{\tau, T}$, by Lemma 3.6, we have the existence of a unique couple $(\tilde{u}^1, \tilde{v}^1) \in \mathcal{U}_{\tau, T} \times \mathcal{V}_{\tau, T}$ such that $\alpha_1(\tilde{v}^1) = \tilde{u}^1$, $\beta_1(\tilde{u}^1) = \tilde{v}^1$. On the other hand, for $\alpha \in \mathcal{A}_{t, T}$ and $\beta \in \mathcal{B}_{t, T}$, there exists a unique couple $(u^*, v^*) \in \mathcal{U}_{t, T} \times \mathcal{V}_{t, T}$ such that $\alpha(v^*) = u^*$, $\beta(u^*) = v^*$. Consequently,

$$\alpha(v^0 \oplus \tilde{v}^1) = \alpha_0(v^0) \oplus \alpha_1(\tilde{v}^1) = u^0 \oplus \tilde{u}^1, \quad \beta(u^0 \oplus \tilde{u}^1) = v^0 \oplus \beta_1(\tilde{u}^1) = v^0 \oplus \tilde{v}^1.$$

From the uniqueness of (u^*, v^*) it follows that $\tilde{u}^1 = u^*|_{[\tau, T]}$, $\tilde{v}^1 = v^*|_{[\tau, T]}$. Therefore,

$${}^iY_\tau^{\tau, x, \alpha_1, \beta_1} = {}^iY_\tau^{\tau, x, \tilde{u}^1, \tilde{v}^1} = {}^iY_\tau^{\tau, x, u^*, v^*} = {}^iY_\tau^{\tau, x, \alpha, \beta}.$$

For $\alpha \in \mathcal{A}_{t,T}$,

$$\text{esssup}_{\bar{\alpha}_1 \in \mathcal{A}_{\tau,T}} \text{essinf}_{\beta_1 \in \mathcal{B}_{\tau,T}} {}^i Y_{\tau}^{\tau,x,\bar{\alpha}_1,\beta_1} \geq \text{essinf}_{\beta_1 \in \mathcal{B}_{\tau,T}} {}^i Y_{\tau}^{\tau,x,\alpha_1,\beta_1} \geq \text{essinf}_{\beta \in \mathcal{B}_{t,T}} {}^i Y_{\tau}^{\tau,x,\alpha,\beta}.$$

The latter estimate takes into account that we associated with β_1 a particular β , $\beta(u) = v^0 \oplus \beta_1(u^1)$, $u^1 = u|_{[\tau,T]}$, independently of α . Consequently,

$$\text{esssup}_{\alpha_1 \in \mathcal{A}_{\tau,T}} \text{essinf}_{\beta_1 \in \mathcal{B}_{\tau,T}} {}^i Y_{\tau}^{\tau,x,\alpha_1,\beta_1} \geq \text{esssup}_{\alpha \in \mathcal{A}_{t,T}} \text{essinf}_{\beta \in \mathcal{B}_{t,T}} {}^i Y_{\tau}^{\tau,x,\alpha,\beta}.$$

Step 2: For all $\varepsilon > 0$, there exists $\alpha_1^\varepsilon \in \mathcal{A}_{\tau,T}$ such that

$$\text{esssup}_{\alpha_1 \in \mathcal{A}_{\tau,T}} \text{essinf}_{\bar{\beta}_1 \in \mathcal{B}_{\tau,T}} {}^i Y_{\tau}^{\tau,x,\alpha_1,\bar{\beta}_1} \leq \text{essinf}_{\bar{\beta}_1 \in \mathcal{B}_{\tau,T}} {}^i Y_{\tau}^{\tau,x,\alpha_1^\varepsilon,\bar{\beta}_1} + \varepsilon. \quad (6.6)$$

For all $\beta \in \mathcal{B}_{t,T}$ and an arbitrary fixed $u^0 \in \mathcal{U}_{t,\tau}$, we define, for $u^1 \in \mathcal{U}_{\tau,T}$ and $v \in \mathcal{V}_{t,T}$,

$$\beta_1(u^1) := \beta(u^0 \oplus u^1)|_{[\tau,T]}, \quad \alpha^\varepsilon(v) := u^0 \oplus \alpha_1^\varepsilon(v^1), \quad v^1 = v|_{[\tau,T]}. \quad (6.7)$$

Then $\alpha^\varepsilon \in \mathcal{A}_{t,T}$ and $\beta_1 \in \mathcal{B}_{\tau,T}$.

Since $\alpha^\varepsilon \in \mathcal{A}_{t,T}$ and $\beta \in \mathcal{B}_{t,T}$, by Lemma 3.6 we have the existence of a unique couple $(u^*, v^*) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$ such that $\alpha^\varepsilon(v^*) = u^*$, $\beta(u^*) = v^*$. On the other hand, from $\alpha_1^\varepsilon \in \mathcal{A}_{\tau,T}$ and $\beta_1 \in \mathcal{B}_{\tau,T}$ it follows that there exists a unique couple $(\tilde{u}^1, \tilde{v}^1) \in \mathcal{U}_{\tau,T} \times \mathcal{V}_{\tau,T}$ such that $\alpha_1^\varepsilon(\tilde{v}^1) = \tilde{u}^1$, $\beta_1(\tilde{u}^1) = \tilde{v}^1$. Since β is nonanticipative, $\beta(u^0 \oplus \tilde{u}^1)|_{[t,\tau]}$ depends only on u^0 and not on \tilde{u}^1 . We put $\beta_0(\bar{u}^0) = \beta(\bar{u}^0 \oplus \tilde{u}^1)|_{[t,\tau]}$, $\bar{u}^0 \in \mathcal{U}_{t,\tau}$. Consequently, $\beta_0 \in \mathcal{B}_{t,\tau}$, and

$$\begin{aligned} \beta(u^0 \oplus \tilde{u}^1) &= \beta(u^0 \oplus \tilde{u}^1)|_{[t,\tau]} \oplus \beta_1(\tilde{u}^1) = \beta_0(u^0) \oplus \beta_1(\tilde{u}^1) = \beta_0(u^0) \oplus \tilde{v}^1, \\ \alpha^\varepsilon(\beta_0(u^0) \oplus \tilde{v}^1) &= u^0 \oplus \alpha_1^\varepsilon(\tilde{v}^1) = u^0 \oplus \tilde{u}^1. \end{aligned}$$

Due to the uniqueness of (u^*, v^*) we obtain $v^* = \beta_0(u^0) \oplus \tilde{v}^1$, $u^* = u^0 \oplus \alpha_1^\varepsilon(\tilde{v}^1) = u^0 \oplus \tilde{u}^1$. Therefore,

$${}^i Y_{\tau}^{\tau,x,\alpha_1^\varepsilon,\beta_1} = {}^i Y_{\tau}^{\tau,x,\tilde{u}^1,\tilde{v}^1} = {}^i Y_{\tau}^{\tau,x,u^*,v^*} = {}^i Y_{\tau}^{\tau,x,\alpha^\varepsilon,\beta},$$

from which combined with (6.6) we get

$$\begin{aligned} \text{esssup}_{\alpha_1 \in \mathcal{A}_{\tau,T}} \text{essinf}_{\bar{\beta}_1 \in \mathcal{B}_{\tau,T}} {}^i Y_{\tau}^{\tau,x,\alpha_1,\bar{\beta}_1} &\leq \text{essinf}_{\bar{\beta}_1 \in \mathcal{B}_{\tau,T}} {}^i Y_{\tau}^{\tau,x,\alpha_1^\varepsilon,\bar{\beta}_1} + \varepsilon \\ &\leq \text{essinf}_{\beta \in \mathcal{B}_{t,T}} {}^i Y_{\tau}^{\tau,x,\alpha^\varepsilon,\beta} + \varepsilon \leq \text{esssup}_{\alpha \in \mathcal{A}_{t,T}} \text{essinf}_{\beta \in \mathcal{B}_{t,T}} {}^i Y_{\tau}^{\tau,x,\alpha,\beta} + \varepsilon. \end{aligned}$$

For the second estimate we used that β_1 in (6.7) is defined with the help of β and, thus, runs only a subclass of $\mathcal{B}_{\tau,T}$. The above both steps allow to conclude the proof. \square

For a stopping time τ with values in $[t, T]$ we define the value functions for a game over the stochastic interval $[[\tau, T]]$:

$$\begin{aligned} \overline{W}_i(\tau, x) &:= \text{esssup}_{\alpha \in \mathcal{A}_{\tau,T}} \text{essinf}_{\beta \in \mathcal{B}_{\tau,T}} {}^i Y_{\tau}^{\tau,x,\alpha,\beta}, \\ \overline{U}_i(\tau, x) &:= \text{essinf}_{\beta \in \mathcal{B}_{\tau,T}} \text{esssup}_{\alpha \in \mathcal{A}_{\tau,T}} {}^i Y_{\tau}^{\tau,x,\alpha,\beta}. \end{aligned}$$

Remark 6.7. Obviously, $\overline{W}_i(t, x) = W_i(t, x)$, $\overline{U}_i(t, x) = U_i(t, x)$, for all $(t, x) \in [0, T] \times \mathbb{R}^n$, and it can be checked in a straight-forward manner that for all discrete valued stopping times τ ($0 \leq \tau \leq T$), $i = 1, 2$,

$$\begin{aligned} \overline{W}_i(\tau, x) &= W_i(\tau, x) (:= W_i(t, x)|_{t=\tau}), \\ \overline{U}_i(\tau, x) &= U_i(\tau, x) (:= U_i(t, x)|_{t=\tau}), \quad \mathbb{P} - a.s. \end{aligned}$$

Our objective is to extend this result to general stopping times. For this end we have to extend Proposition 3.9 and Theorem 6.1.

Similar to Proposition 3.9 we have from standard BSDEs estimates the following proposition.

Proposition 6.8. *There exists a constant $C > 0$ such that, for all stopping time τ ($0 \leq \tau \leq T$), $x, x' \in \mathbb{R}^n$, $i = 1, 2$,*

$$|\overline{W}_i(\tau, x) - \overline{W}_i(\tau, x')| \leq C|x - x'|, \quad |\overline{W}_i(\tau, x)| \leq C(1 + |x|).$$

The same property holds for \overline{U}_i .

Theorem 6.9. *Let the assumptions (H4) and (H5) hold. Then the following dynamic programming principles hold: For any stopping times τ, η with $0 \leq t < \tau \leq \eta \leq T$, $x \in \mathbb{R}^n$, $i = 1, 2$,*

$$\begin{aligned} \overline{W}_i(\tau, x) &= \operatorname{esssup}_{\alpha \in \mathcal{A}_{\tau, \eta}} \operatorname{essinf}_{\beta \in \mathcal{B}_{\tau, \eta}} {}^i G_{\tau, \eta}^{t, x; \alpha, \beta} [\overline{W}_{N_{\eta}^{\tau, i}}(\eta, X_{\eta}^{\tau, x; \alpha, \beta})], \\ \overline{U}_i(\tau, x) &= \operatorname{essinf}_{\beta \in \mathcal{B}_{\tau, \eta}} \operatorname{esssup}_{\alpha \in \mathcal{A}_{\tau, \eta}} {}^i G_{\tau, \eta}^{t, x; \alpha, \beta} [\overline{U}_{N_{\eta}^{\tau, i}}(\eta, X_{\eta}^{\tau, x; \alpha, \beta})]. \end{aligned} \quad (6.8)$$

Proof. We only comment the proof of the first relation. Let us denote by $\widehat{W}_i(\tau, x)$ the right hand side of 6.8. We prove in a straight-forward way that $\overline{W}_i(\tau, x) = \widehat{W}_i(\tau, x)$. For this we adapt the proof of the Lemmas 6.3 and 6.4 in an obvious manner. \square

From Theorem 6.9 and Proposition 6.8 we can deduce the following proposition.

Proposition 6.10. *Let the assumptions (H4) and (H5) be satisfied. Then, for any stopping time τ, η with $0 \leq t < \tau \leq \eta \leq T$, $x \in \mathbb{R}^n$, where η is supposed to be $\sigma(\tau)$ -measurable, we have the following:*

$$|\overline{W}_i(\tau, x) - \overline{W}_i(\eta, x)| \leq C(1 + |x|)|\tau - \eta|^{\frac{1}{2}}.$$

Proof. We only prove that $\overline{W}_i(\tau, x) - \overline{W}_i(\eta, x) \leq C(1 + |x|)\tau - \eta|^{\frac{1}{2}}$, since the other inequality can be proved in a similar manner. Let $\varepsilon > 0$. In analogy to (6.3), but now for (τ, η) instead of $(t, t + \delta)$, we can show that there exists some $\alpha^\varepsilon \in \mathcal{A}_{\tau, \eta}$ such that, for any $\beta \in \mathcal{B}_{\tau, \eta}$,

$$\overline{W}_i(\tau, x) \leq {}^i G_{\tau, \eta}^{\tau, x; \alpha^\varepsilon, \beta} [\overline{W}_{N_{\eta}^{\tau, i}}(\eta, X_{\eta}^{\tau, x; \alpha^\varepsilon, \beta})] + \varepsilon.$$

Consequently, $\overline{W}_i(\tau, x) - \overline{W}_i(\eta, x) \leq I_1 + I_2 + I_3 + \varepsilon$, where

$$\begin{aligned} I_1 &:= {}^i G_{\tau, \eta}^{\tau, x; \alpha^\varepsilon, \beta} [\overline{W}_{N_{\eta}^{\tau, i}}(\eta, X_{\eta}^{\tau, x; \alpha^\varepsilon, \beta})] - {}^i G_{\tau, \eta}^{\tau, x; \alpha^\varepsilon, \beta} [\overline{W}_{N_{\eta}^{\tau, i}}(\eta, x)], \\ I_2 &:= {}^i G_{\tau, \eta}^{\tau, x; \alpha^\varepsilon, \beta} [\overline{W}_{N_{\eta}^{\tau, i}}(\eta, x)] - {}^i G_{\tau, \eta}^{\tau, x; \alpha^\varepsilon, \beta} [\overline{W}_i(\eta, x)], \\ I_3 &:= {}^i G_{\tau, \eta}^{\tau, x; \alpha^\varepsilon, \beta} [\overline{W}_i(\eta, x)] - \overline{W}_i(\eta, x). \end{aligned}$$

Thanks to Lemma 3.6, we let $(u, v) \in \mathcal{U}_{\tau, \eta} \times \mathcal{V}_{\tau, \eta}$ be such that $\alpha^\varepsilon(v) = u, \beta(u) = v$, on $[[\tau, \eta]]$.

By virtue of Proposition 6.8 there exists some positive constant C which does not depend on α^ε and β such that

$$\begin{aligned} |I_1| &\leq C(\mathbb{E}[|\overline{W}_{N_{\eta}^{\tau, i}}(\eta, X_{\eta}^{\tau, x; u, v}) - \overline{W}_{N_{\eta}^{\tau, i}}(\eta, x)|^2 | \mathcal{F}_\tau])^{\frac{1}{2}} \\ &\leq C(\mathbb{E}[|X_{\eta}^{\tau, x; u, v} - x|^2 | \mathcal{F}_\tau])^{\frac{1}{2}} \leq C(1 + |x|^2)|\tau - \eta|^{\frac{1}{2}}, \end{aligned}$$

where the latter relation follows from standard SDE estimates. Let us denote by $({}^i Y, {}^i Z, {}^i H)$ the solution of the BSDE:

$$\begin{cases} -d {}^i Y_s &= f_{N_s^{\tau, i}}(s, X_s^{\tau, x; u, v}, {}^i Y_s, {}^i H_s, {}^i Z_s, u_s, v_s) ds \\ &\quad - \lambda {}^i H_s ds - {}^i Z_s dB_s - {}^i H_s d\tilde{N}_s, \quad s \in [\tau, \eta], \\ {}^i Y_\eta &= \overline{W}_i(\eta, x). \end{cases}$$

From the definition of our backward stochastic semigroup and the $\sigma(\tau)$ -measurability of η it follows that

$$\begin{aligned}
I_3 &= {}^i G_{\tau, \eta}^{t, x; \alpha^\varepsilon, \beta} [\overline{W}_i(\eta, x)] - \overline{W}_i(\eta, x) \\
&= \mathbb{E}[\overline{W}_i(\eta, x) + \int_{\tau}^{\eta} f_{N_s^{\tau, i}}(s, X_s^{\tau, x; u, v}, {}^i Y_s, {}^i H_s, {}^i Z_s, u_s, v_s) ds \\
&\quad - \lambda \int_{\tau}^{\eta} {}^i H_s ds - \int_{\tau}^{\eta} {}^i Z_s dB_s - \int_{\tau}^{\eta} {}^i H_s d\tilde{N}_s \mid \mathcal{F}_{\tau}] - \overline{W}_i(\eta, x) \\
&= \mathbb{E}[\int_{\tau}^{\eta} f_{N_s^{\tau, i}}(s, X_s^{\tau, x; u, v}, {}^i Y_s, {}^i H_s, {}^i Z_s, u_s, v_s) ds - \lambda \int_{\tau}^{\eta} {}^i H_s ds \mid \mathcal{F}_{\tau}].
\end{aligned}$$

Therefore, the Schwartz inequality and the Appendix of [9] yield

$$\begin{aligned}
|I_3| &\leq \mathbb{E}[\int_{\tau}^{\eta} (|f_{N_s^{\tau, i}}(s, X_s^{\tau, x; u, v}, {}^i Y_s, {}^i H_s, {}^i Z_s, u_s, v_s)| + \lambda |{}^i H_s|) ds \mid \mathcal{F}_{\tau}] \\
&\leq |\tau - \eta|^{\frac{1}{2}} \mathbb{E}[\int_{\tau}^{\eta} (|f_{N_s^{\tau, i}}(s, X_s^{\tau, x; u, v}, {}^i Y_s, {}^i H_s, {}^i Z_s, u_s, v_s)| + \lambda |{}^i H_s|)^2 ds \mid \mathcal{F}_{\tau}]^{\frac{1}{2}} \\
&\leq C |\tau - \eta|^{\frac{1}{2}} \mathbb{E}[\int_{\tau}^{\eta} (|f_{N_s^{\tau, i}}(s, X_s^{\tau, x; u, v}, 0, 0, 0, u_s, v_s)|^2 + |{}^i Y_s|^2 + |{}^i H_s|^2 + |{}^i Z_s|^2) ds \mid \mathcal{F}_{\tau}]^{\frac{1}{2}} \\
&\leq C |\tau - \eta|^{\frac{1}{2}} \mathbb{E}[\int_{\tau}^{\eta} (1 + |X_s^{\tau, x; u, v}|^2 + |{}^i Y_s|^2 + |{}^i H_s|^2 + |{}^i Z_s|^2) ds \mid \mathcal{F}_{\tau}]^{\frac{1}{2}} \\
&\leq C(1 + |x|) |\tau - \eta|^{\frac{1}{2}}.
\end{aligned}$$

Finally, let us give the estimate of I_2 . From Lemma 2.3 we have

$$\begin{aligned}
|I_2| &\leq C \mathbb{E}[|\overline{W}_{N_{\eta}^{\tau, i}}(\eta, x) - \overline{W}_i(\eta, x)| \mid \mathcal{F}_{\tau}] = C \mathbb{E}[|\overline{W}_{N_{\eta}^{\tau, i}}(\eta, x) - \overline{W}_i(\eta, x)| 1_{\{N_{\eta}^{\tau, i} \neq i\}} \mid \mathcal{F}_{\tau}] \\
&\leq C(1 + |x|) \mathbb{E}[1_{\{N_{\eta}^{\tau, i} \neq i\}} \mid \mathcal{F}_{\tau}] = C(1 + |x|) \mathbb{P}[\{N_{\eta}^{\tau, i} \neq i\} \mid \mathcal{F}_{\tau}] \\
&\leq C(1 + |x|)(1 - \exp(-\lambda(\eta - \tau))) \leq C(1 + |x|) |\tau - \eta|.
\end{aligned}$$

Consequently, from the above inequalities we deduce that

$$\overline{W}_i(\tau, x) - \overline{W}_i(\eta, x) \leq C(1 + |x|) |\tau - \eta|^{\frac{1}{2}} + \varepsilon, \quad \varepsilon > 0,$$

from where we conclude

$$\overline{W}_i(\tau, x) - \overline{W}_i(\eta, x) \leq C(1 + |x|) |\tau - \eta|^{\frac{1}{2}}.$$

The desired result then follows. \square

Proposition 6.11. *Under the assumptions (H4) and (H5), the following holds: $\overline{W}_i(\tau, x) = W_i(\tau, x)$, i.e.,*

$$W_i(\tau, x) = \text{esssup}_{\alpha \in \mathcal{A}_{\tau, T}} \text{essinf}_{\beta \in \mathcal{B}_{\tau, T}} {}^i Y_{\tau}^{\tau, x, \alpha, \beta},$$

for all stopping time τ with values in $[t, T]$.

Proof. Let $t \in [0, T]$ be such that $t \leq \tau \leq T$. Let us define, for $i = 1, \dots, 2^n$,

$$t_i = \frac{i(T - t)}{2^n} + t, \quad A_i = \left\{ t_{i-1} < \tau \leq t_i \right\}, \quad \text{and } A_0 = \{\tau = t\}, \quad \tau_n = \sum_{i=0}^{2^n} t_i 1_{A_i}.$$

Obviously, $0 \leq \tau_n - \tau \leq \frac{1}{2^n}$, and from the definition of W_i as well as that of the essential infimum and supremum of a family of random variables we deduce

$$W_i(\tau_n, x) (= W_i(s, x)|_{s=\tau_n}) = \text{esssup}_{\alpha \in \mathcal{A}_{t,T}} \text{essinf}_{\beta \in \mathcal{B}_{t,T}} {}^i Y_{\tau_n}^{\tau_n, x, \alpha, \beta}.$$

Therefore, $W_i(\tau_n, x) = \overline{W}_i(\tau_n, x)$, \mathbb{P} -a.s. Since $\tau \leq \tau_n \leq T$, and τ_n is $\sigma(\tau)$ -measurable, it follows from the Propositions 6.5 and 6.10 that

$$|\overline{W}_i(\tau, x) - \overline{W}_i(\tau_n, x)| \leq C(1 + |x|)|\tau - \tau_n|^{\frac{1}{2}} \rightarrow 0, \text{ as } n \rightarrow \infty,$$

and

$$|W_i(\tau, x) - W_i(\tau_n, x)| \leq C(1 + |x|)|\tau - \tau_n|^{\frac{1}{2}} \rightarrow 0, \text{ as } n \rightarrow \infty,$$

from where we conclude that also $W_i(\tau, x) = \overline{W}_i(\tau, x)$. Therefore,

$$W_i(\tau, x) = \text{esssup}_{\alpha \in \mathcal{A}_{\tau,T}} \text{essinf}_{\beta \in \mathcal{B}_{\tau,T}} {}^i Y_{\tau}^{\tau, x, \alpha, \beta}.$$

The proof is complete. \square

From the above proposition and Proposition 6.10 we immediately have:

Proposition 6.12. *Under the assumptions (H4) and (H5), there is some positive constant C such that, for any stopping times τ, η with $0 \leq t < \tau \leq \eta \leq T$, $x \in \mathbb{R}^n$, where η is supposed to be $\sigma(\tau)$ -measurable, we have the following:*

$$|W_i(\tau, x) - W_i(\eta, x)| \leq C(1 + |x|)|\tau - \eta|^{\frac{1}{2}}.$$

Theorem 3.10 follows from Proposition 6.11 and Theorem 6.9. We also have the following statement, which is a direct consequence of Proposition 6.11.

Proposition 6.13. *Under the assumptions (H4) and (H5), the following holds:*

$$W_{N_{\tau}^{t,i}}(\tau, x) = \text{esssup}_{\alpha \in \mathcal{A}_{\tau,T}} \text{essinf}_{\beta \in \mathcal{B}_{\tau,T}} N_{\tau}^{t,i} Y_{\tau}^{\tau, x, \alpha, \beta}.$$

6.2 Proof of Theorem 4.2

Proof. We only give the proof for $U = (U_1, U_2)$, that for $W = (W_1, W_2)$ uses a similar argument. Let $i = 1, 2$ be arbitrarily fixed, $(t, x) \in [0, T] \times \mathbb{R}^n$, and $\delta > 0$. We put $\tau^\delta = \inf\{s \geq t, N_s^{t,i} \neq i\} \wedge (t + \delta)$.

For $\varphi \in C_{l,b}^3([0, T] \times \mathbb{R}^n)$, we define

$$\begin{aligned} F(s, x, y, h, z, u, v) &= \frac{\partial}{\partial s} \varphi(s, x) + \frac{1}{2} \text{tr}(\sigma \sigma^T(s, x, u, v) D^2 \varphi(s, x)) + D\varphi(s, x) b(s, x, u, v) \\ &\quad + f_i(s, x, y + \varphi(s, x), h + U_{m(i+1)}(s, x) - \varphi(s, x), z + D\varphi(s, x) \cdot \sigma(s, x, u, v), u, v), \end{aligned}$$

where $(s, x, y, h, z, u, v) \in [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times U \times V$.

Let us consider the following BSDE on the interval $[t, \tau^\delta]$:

$$\begin{cases} -dY_s^{1,u,v} = F(s, X_s^{t,x;u,v}, Y_s^{1,u,v}, H_s^{1,u,v}, Z_s^{1,u,v}, u_s, v_s) ds - \lambda H_s^{1,u,v} ds \\ \quad - Z_s^{1,u,v} dB_s - H_s^{1,u,v} d\tilde{N}_s, \\ Y_{\tau^\delta}^{1,u,v} = 0, \quad s \in [t, \tau^\delta], \end{cases} \quad (6.9)$$

where $X^{t,x;u,v}$ is the solution of (3.1) with $\zeta = x \in \mathbb{R}^n$.

We notice that $F(s, x, y, h, z, u, v)$ is Lipschitz in (y, h, z) , uniformly in (s, x, u, v) , and there exists a positive constant C such that

$$F(s, x, 0, 0, 0, u, v) \leq C(1 + |x|^2), \quad (s, x, u, v) \in [0, T] \times \mathbb{R}^n \times U \times V.$$

Consequently, BSDE (6.9) has a unique solution $(Y^{1,u,v}, H^{1,u,v}, Z^{1,u,v})$. This solution obviously depends on δ , but for the sake of simplifying the notations we don't add δ as superscript to $(Y^{1,u,v}, H^{1,u,v}, Z^{1,u,v})$.

For the proof of Theorem 4.2, which extends the approaches in [6] and [8] to SDGs with jumps, we admit the following lemmas for the moment, they will be proven after.

Lemma 6.14. *For every $s \in [t, \tau^\delta]$, the following holds:*

$$Y_{s \wedge \tau^\delta}^{1,u,v} = {}^i G_{s \wedge \tau^\delta, \tau^\delta}^{t,x;u,v} \left[\varphi(\tau^\delta, X_{\tau^\delta}^{t,x;u,v}) 1_{N_{\tau^\delta}^{t,i}=i} + U_{m(i+1)}(\tau^\delta, X_{\tau^\delta}^{t,x;u,v}) 1_{N_{\tau^\delta}^{t,i}=m(i+1)} \right] - \bar{Y}_{s \wedge \tau^\delta},$$

where

$$\bar{Y}_s = \varphi(s, X_s^{t,x;u,v}) + \int_t^s \left(U_{m(i+1)}(r, X_r^{t,x;u,v}) - \varphi(r, X_r^{t,x;u,v}) \right) dN_r.$$

In particular,

$$Y_t^{1,u,v} = {}^i G_{t, \tau^\delta}^{t,x;u,v} \left[\varphi(\tau^\delta, X_{\tau^\delta}^{t,x;u,v}) 1_{N_{\tau^\delta}^{t,i}=i} + U_{m(i+1)}(\tau^\delta, X_{\tau^\delta}^{t,x;u,v}) 1_{N_{\tau^\delta}^{t,i}=m(i+1)} \right] - \varphi(t, x).$$

(Recall that $m(j) = 1$, if j is odd, and $m(j) = 2$, if j is even.)

We also consider the following BSDE where, in equation (6.9), $X^{t,x;u,v}$ is substituted by its deterministic initial value x :

$$\begin{cases} -dY_s^{2,u,v} = F(s, x, Y_s^{2,u,v}, H_s^{2,u,v}, Z_s^{2,u,v}, u_s, v_s) ds - \lambda H_s^{2,u,v} ds \\ \quad - Z_s^{2,u,v} dB_s - H_s^{2,u,v} d\tilde{N}_s, \\ Y_{\tau^\delta}^{2,u,v} = 0, \quad s \in [t, \tau^\delta]. \end{cases} \quad (6.10)$$

Then we have the following comparison between $Y^{1,u,v}$ and $Y^{2,u,v}$.

Lemma 6.15. *For every $\delta \in (0, 1)$ and $u \in \mathcal{U}_{t,t+\delta}, v \in \mathcal{V}_{t,t+\delta}$,*

$$|Y_t^{1,u,v} - Y_t^{2,u,v}| \leq C\delta^{\frac{3}{2}}, \quad \mathbb{P} - a.s.,$$

where the constant C does not depend on the control processes u and v , neither on $\delta > 0$.

Moreover, we have

Lemma 6.16. *For every $u \in \mathcal{U}_{t,t+\delta}, v \in \mathcal{V}_{t,t+\delta}$, we have*

$$\mathbb{E} \left[\int_t^{t+\delta} |Y_s^{2,u,v}| ds | \mathcal{F}_t \right] + \mathbb{E} \left[\int_t^{t+\delta} |Z_s^{2,u,v}| ds | \mathcal{F}_t \right] + \mathbb{E} \left[\int_t^{t+\delta} |H_s^{2,u,v}| ds | \mathcal{F}_t \right] \leq C\delta^{\frac{3}{2}}, \quad \mathbb{P} - a.s.,$$

where the constant C is independent of t , δ as well as of the control processes u, v .

Let us prove that $U = (U_1, U_2)$ is a viscosity solution of the system (4.1). We begin with showing that

a) $U = (U_1, U_2)$ is a viscosity subsolution of the system (4.1).

Let $i = 1, 2$ be arbitrarily fixed, and we suppose that $U_i \leq \varphi$ and $U_i(t, x) = \varphi(t, x)$. We claim that

$$\inf_{v \in V} \sup_{u \in U} F(t, x, 0, 0, 0, u, v) \geq 0.$$

Observe that, this claim just means that $U = (U_1, U_2)$ satisfies (4.3) of Definition 4.1; we will come back to this point. We make the proof by contradiction, and suppose that the above claim is not true. Then, thanks to the continuity of F , there exist some $\theta > 0, v^* \in V$ and $0 < \delta' \leq T - t$ such that

$$\sup_{u \in U} F(s, x, 0, 0, 0, u, v^*) \leq -\theta < 0, \quad (6.11)$$

for all $s \in [t, t + \delta']$. Since $U_i \leq \varphi$ and $U_i(t, x) = \varphi(t, x)$, we have

$$\begin{aligned} & \varphi(\tau^\delta, X_{\tau^\delta}^{t,x;\alpha,\beta}) 1_{N_{\tau^\delta}^{t,i}=i} + U_{m(i+1)}(\tau^\delta, X_{\tau^\delta}^{t,x;\alpha,\beta}) 1_{N_{\tau^\delta}^{t,i}=m(i+1)} \\ & \geq U_i(\tau^\delta, X_{\tau^\delta}^{t,x;\alpha,\beta}) 1_{N_{\tau^\delta}^{t,i}=i} + U_{m(i+1)}(\tau^\delta, X_{\tau^\delta}^{t,x;\alpha,\beta}) 1_{N_{\tau^\delta}^{t,i}=m(i+1)} \\ & = U_{N_{\tau^\delta}^{t,i}}(\tau^\delta, X_{\tau^\delta}^{t,x;\alpha,\beta}). \end{aligned}$$

From the Lemmas 2.2 and 6.14 it follows that

$$\begin{aligned} Y_t^{1,\alpha,\beta} &= {}^i G_{t,\tau^\delta}^{t,x;\alpha,\beta} \left[\varphi(\tau^\delta, X_{\tau^\delta}^{t,x;\alpha,\beta}) 1_{N_{\tau^\delta}^{t,i}=i} + U_{m(i+1)}(\tau^\delta, X_{\tau^\delta}^{t,x;\alpha,\beta}) 1_{N_{\tau^\delta}^{t,i}=m(i+1)} \right] - \varphi(t, x) \\ &\geq {}^i G_{t,\tau^\delta}^{t,x;\alpha,\beta} [U_{N_{\tau^\delta}^{t,i}}(\tau^\delta, X_{\tau^\delta}^{t,x;\alpha,\beta})] - \varphi(t, x). \end{aligned}$$

On the other hand, by virtue of Theorem 3.10 we have

$$\text{essinf}_{\beta \in \mathcal{B}_{t,\tau^\delta}} \text{esssup}_{u \in \mathcal{U}_{t,\tau^\delta}} {}^i G_{t,\tau^\delta}^{t,x;\alpha,\beta} [U_{N_{\tau^\delta}^{t,i}}(\tau^\delta, X_{\tau^\delta}^{t,x;\alpha,\beta})] - \varphi(t, x) = U_i(t, x) - \varphi(t, x) = 0.$$

Therefore, $\text{essinf}_{\beta \in \mathcal{B}_{t,\tau^\delta}} \text{esssup}_{\alpha \in \mathcal{A}_{t,\tau^\delta}} Y_t^{1,\alpha,\beta} \geq 0$. Putting $\beta^*(u) = v^*$ on $[t, \tau^\delta] \times \mathcal{U}_{t,\tau^\delta}$, we have $\beta^* \in \mathcal{B}_{t,\tau^\delta}$. From Lemma 6.15 it follows that

$$\begin{aligned} 0 &\leq \text{essinf}_{\beta \in \mathcal{B}_{t,\tau^\delta}} \text{esssup}_{\alpha \in \mathcal{A}_{t,\tau^\delta}} Y_t^{1,\alpha,\beta} \\ &\leq \text{esssup}_{\alpha \in \mathcal{A}_{t,\tau^\delta}} Y_t^{1,\alpha,\beta^*} \\ &\leq \text{esssup}_{\alpha \in \mathcal{A}_{t,\tau^\delta}} Y_t^{2,\alpha,\beta^*} + C\delta^{\frac{3}{2}} \\ &= \text{esssup}_{\alpha \in \mathcal{A}_{t,\tau^\delta}} Y_t^{2,\alpha(v^*),v^*} + C\delta^{\frac{3}{2}} \\ &\leq \text{esssup}_{u \in \mathcal{U}_{t,\tau^\delta}} Y_t^{2,u,v^*} + C\delta^{\frac{3}{2}}. \end{aligned}$$

Here we have used that $Y_t^{2,\alpha,\beta^*} = Y_t^{2,\alpha(v^*),v^*}$, since $\beta^*(\alpha(v^*)) = v^*$. Furthermore, for all $\delta > 0, \varepsilon > 0$, there exists $u^\varepsilon \in \mathcal{U}_{t,\tau^\delta}$ such that

$$Y_t^{2,u^\varepsilon,v^*} \geq \text{esssup}_{u \in \mathcal{U}_{t,\tau^\delta}} Y_t^{2,u,v^*} - \varepsilon\delta.$$

The two above inequalities yield

$$Y_t^{2,u^\varepsilon,v^*} \geq -C\delta^{\frac{3}{2}} - \varepsilon\delta. \quad (6.12)$$

Since

$$Y_t^{2,u^\varepsilon,v^*} = \mathbb{E} \left[\int_t^{\tau^\delta} F(s, x, Y_s^{2,u^\varepsilon,v^*}, H_s^{2,u^\varepsilon,v^*}, Z_s^{2,u^\varepsilon,v^*}, u_s^\varepsilon, v_s^*) ds \middle| \mathcal{F}_t \right],$$

we can deduce from the Lipschitz property of F in (y, h, z) that

$$Y_t^{2,u^\varepsilon,v^*} = \mathbb{E} \left[\int_t^{\tau^\delta} F(s, x, Y_s^{2,u^\varepsilon,v^*}, H_s^{2,u^\varepsilon,v^*}, Z_s^{2,u^\varepsilon,v^*}, u_s^\varepsilon, v_s^*) ds \middle| \mathcal{F}_t \right]$$

$$\begin{aligned}
&\leq \mathbb{E}\left[\int_t^{\tau^\delta} F(s, x, 0, 0, 0, u_s^\varepsilon, v_s^*) ds \middle| \mathcal{F}_t\right] \\
&\quad + C\mathbb{E}\left[\int_t^{\tau^\delta} \left(|Y_s^{2, u^\varepsilon, v^*}| + |H_s^{2, u^\varepsilon, v^*}| + |Z_s^{2, u^\varepsilon, v^*}| \right) ds \middle| \mathcal{F}_t\right]. \tag{6.13}
\end{aligned}$$

We notice that

$$\begin{aligned}
\mathbb{E}[t + \delta - \tau^\delta | \mathcal{F}_t] &\leq \delta \mathbb{E}[1_{t+\delta > \tau^\delta} | \mathcal{F}_t] = \delta \mathbb{P}[t + \delta > \tau^\delta | \mathcal{F}_t] \\
&= \delta(1 - \exp(-\lambda\delta)) \leq \lambda\delta^2.
\end{aligned}$$

Choosing $\delta < \delta'$ we obtain from (6.11)

$$\begin{aligned}
\mathbb{E}\left[\int_t^{\tau^\delta} F(s, x, 0, 0, 0, u_s^\varepsilon, v_s^*) ds \middle| \mathcal{F}_t\right] &\leq -\theta \mathbb{E}[\tau^\delta - t | \mathcal{F}_t] \\
&\leq \theta(\mathbb{E}[t + \delta - \tau^\delta | \mathcal{F}_t] - \delta) \leq \theta(\lambda\delta^2 - \delta).
\end{aligned}$$

This together with (6.12), (6.13) and Lemma 6.16 yields $-C\delta^{\frac{3}{2}} - \varepsilon\delta \leq \lambda\delta^2\theta - \theta\delta$. Therefore, $-C\delta^{\frac{1}{2}} - \varepsilon \leq \lambda\delta\theta - \theta$. Letting $\delta \downarrow 0$, and $\varepsilon \downarrow 0$ we deduce that $\theta \leq 0$, which is in contradiction with $\theta > 0$. Consequently,

$$\inf_{v \in V} \sup_{u \in U} F(t, x, 0, 0, 0, u, v) \geq 0.$$

Finally, taking into account the definition of F , we see that

$$\begin{aligned}
&\inf_{v \in V} \sup_{u \in U} F(t, x, 0, 0, 0, u, v) \\
&= \frac{\partial}{\partial t} \varphi(t, x) + \inf_{v \in V} \sup_{u \in U} \left\{ \frac{1}{2} \text{tr}(\sigma \sigma^T(t, x, u, v) D^2 \varphi(t, x)) + D\varphi(t, x) b(t, x, u, v) \right. \\
&\quad \left. + f_i(t, x, \varphi(t, x), U_{m(i+1)}(t, x) - \varphi(t, x), D\varphi(t, x) \sigma(t, x, u, v), u, v) \right\} \\
&= \frac{\partial}{\partial t} \varphi(t, x) + \inf_{v \in V} \sup_{u \in U} \left\{ \frac{1}{2} \text{tr}(\sigma \sigma^T(t, x, u, v) D^2 \varphi(t, x)) + D\varphi(t, x) b(t, x, u, v) \right. \\
&\quad \left. + \tilde{f}_i(t, x, U_1(t, x), U_2(t, x), D\varphi(t, x) \sigma(t, x, u, v), u, v) \right\},
\end{aligned}$$

from where it follows that $U = (U_1, U_2)$ is a viscosity subsolution of the system (4.1).

b) Let us show that $U = (U_1, U_2)$ is also a viscosity supersolution of the system (4.1).

Let $i = 1, 2$ be arbitrarily fixed, and we suppose that the test function φ is such that $U_i \geq \varphi$ and $U_i(t, x) = \varphi(t, x)$. Then

$$\begin{aligned}
&\varphi(\tau^\delta, X_{\tau^\delta}^{t, x; \alpha, \beta}) 1_{N_{\tau^\delta}^{t, i} = i} + U_{m(i+1)}(\tau^\delta, X_{\tau^\delta}^{t, x; \alpha, \beta}) 1_{N_{\tau^\delta}^{t, i} = m(i+1)} \\
&\leq U_i(\tau^\delta, X_{\tau^\delta}^{t, x; \alpha, \beta}) 1_{N_{\tau^\delta}^{t, i} = i} + U_{m(i+1)}(\tau^\delta, X_{\tau^\delta}^{t, x; \alpha, \beta}) 1_{N_{\tau^\delta}^{t, i} = m(i+1)} \\
&= U_{N_{\tau^\delta}^{t, i}}(\tau^\delta, X_{\tau^\delta}^{t, x; \alpha, \beta}),
\end{aligned}$$

and thanks to Lemma 6.14 we have

$$\begin{aligned}
Y_t^{1, \alpha, \beta} &= {}^i G_{t, \tau^\delta}^{t, x; \alpha, \beta} [\varphi(\tau^\delta, X_{\tau^\delta}^{t, x; \alpha, \beta}) 1_{N_{\tau^\delta}^{t, i} = i} + U_{m(i+1)}(\tau^\delta, X_{\tau^\delta}^{t, x; \alpha, \beta}) 1_{N_{\tau^\delta}^{t, i} = m(i+1)}] - \varphi(t, x) \\
&\leq {}^i G_{t, \tau^\delta}^{t, x; \alpha, \beta} [U_{N_{\tau^\delta}^{t, i}}(\tau^\delta, X_{\tau^\delta}^{t, x; \alpha, \beta})] - \varphi(t, x).
\end{aligned}$$

Consequently, from Theorem 3.10

$$\operatorname{ess\,inf}_{\beta \in \mathcal{B}_{t,\tau^\delta}} \operatorname{ess\,sup}_{u \in \mathcal{U}_{t,\tau^\delta}} {}^i G_{t,\tau^\delta}^{t,x;\alpha,\beta} [U_{N_{\tau^\delta}^{t,i}}(\tau^\delta, X_{\tau^\delta}^{t,x;\alpha,\beta})] - \varphi(t, x) = 0,$$

we obtain that $\operatorname{ess\,inf}_{\beta \in \mathcal{B}_{t,\tau^\delta}} \operatorname{ess\,sup}_{\alpha \in \mathcal{A}_{t,\tau^\delta}} Y_t^{1,\alpha,\beta} \leq 0$. Then, from Lemma 6.15 it follows that

$$\operatorname{ess\,inf}_{\beta \in \mathcal{B}_{t,\tau^\delta}} \operatorname{ess\,sup}_{\alpha \in \mathcal{A}_{t,\tau^\delta}} Y_t^{2,\alpha,\beta} \leq C\delta^{\frac{3}{2}}.$$

Hence, there exists $\beta^* \in \mathcal{B}_{t,\tau^\delta}$ (depending on δ) such that

$$\operatorname{ess\,sup}_{\alpha \in \mathcal{A}_{t,\tau^\delta}} Y_t^{2,\alpha,\beta^*} \leq 2C\delta^{\frac{3}{2}}. \quad (6.14)$$

From $\beta^* \in \mathcal{B}_{t,\tau^\delta}$ it follows that there exists an increasing sequence of stopping times $\{S_n(u)\}_{n \geq 1}$, for all $u \in \mathcal{U}_{t,\tau^\delta}$, with $t = S_0(u) \leq S_1(u) \leq \dots \leq S_n(u) \leq \dots \leq \tau^\delta$ and $\bigcup_{n \geq 1} \{S_n(u) = \tau^\delta\} = \Omega$, \mathbb{P} -a.s., such that, for all $n \geq 1$ and $u, u' \in \mathcal{U}_{t,\tau^\delta}$ with $u = u'$ on $[[t, S_{n-1}(u)]]$, it holds

$$S_l(u) = S_l(u'), 1 \leq l \leq n, \text{ and } \beta^*(u) = \beta^*(u'), \text{ on } [[t, S_n(u)]].$$

Therefore, S_1 as well as $v^* := \beta^*(u)$ on $[[t, S_1(u)]]$ do not depend on the choice of $u \in \mathcal{U}_{t,\tau^\delta}$. Let us define u^* on $[[t, S_1(u)]]$ as the process such that $u^*_{\cdot \wedge S_1} \in \mathcal{U}_{t,\tau^\delta}$ and

$$F(s, x, 0, 0, 0, u_s^*, v_s^*) = \sup_{u \in \mathcal{U}_{t,\tau^\delta}} F(s, x, 0, 0, 0, u, v_s^*), \quad s \in [t, S_1(u^*)].$$

Putting $v^* := \beta^*(u^*_{\cdot \wedge S_1})$ on $[[t, S_1(u^*)]]$ (Observe that $S_2(u^*)$ only depends on $u^*_{\cdot \wedge S_1}$), let us define the process u^* on $[[t, S_2(u^*)]]$ by $u^*_{(\cdot \wedge S_2(u^*)) \vee S_1(u^*)} \in \mathcal{U}_{t,\tau^\delta}$ such that

$$F(s, x, 0, 0, 0, u_s^*, v_s^*) = \sup_{u \in U} F(s, x, 0, 0, 0, u, v_s^*), \quad s \in [S_1(u^*), S_2(u^*)].$$

Therefore,

$$F(s, x, 0, 0, 0, u_s^*, \beta^*(u^*_{\cdot \wedge S_2(u^*)})_s) = \sup_{u \in U} F(s, x, 0, 0, 0, u, v_s^*), \quad s \in [t, S_2(u^*)].$$

Iterating the above argument we obtain u^* on $[[t, S_\infty[[$, $S_\infty := \lim_{n \rightarrow \infty} \uparrow S_n(u^*) \leq \tau^\delta$. Choosing an arbitrary $u_0 \in U$, we define $u^* := u^* 1_{[t, S_\infty[} + u_0 1_{[S_\infty, \tau^\delta]}$ $\in \mathcal{U}_{t,\tau^\delta}$. Since $\bigcup_{n \geq 1} \{S_n(u^*) = \tau^\delta\} = \Omega$ we can conclude

$$\begin{aligned} F(s, x, 0, 0, 0, u_s^*, \beta^*(u^*_{\cdot \wedge S_n(u^*)})_s) &= \sup_{u \in U} F(s, x, 0, 0, 0, u, v_s^*) \\ &\geq \inf_{v \in V} \sup_{u \in U} F(s, x, 0, 0, 0, u, v), \quad s \in [t, S_n(u^*)]. \end{aligned}$$

for all $n \geq 1$, and hence, that

$$F(s, x, 0, 0, 0, u_s^*, \beta^*(u^*)_s) \geq \inf_{v \in U} \sup_{u \in U} F(s, x, 0, 0, 0, u, v), \quad s \in [t, \tau^\delta]. \quad (6.15)$$

On the other hand, defining $\alpha^*(v) = u^*$ on $[t, \tau^\delta] \times U_{t,\tau^\delta}$, we deduce from (6.14)

$$2C\delta^{\frac{3}{2}} \geq Y_t^{2,\alpha^*,\beta^*} = Y_t^{2,u^*,\beta^*(u^*)}. \quad (6.16)$$

Let us consider the following BSDEs:

$$\begin{cases} -dY_s^{2,u^*,\beta^*(u^*)} = F(s, x, Y_s^{2,u^*,\beta^*(u^*)}, H_s^{2,u^*,\beta^*(u^*)}, Z_s^{2,u^*,\beta^*(u^*)}, u_s^*, \beta^*(u_s^*)) ds \\ \quad - \lambda H_s^{2,u^*,\beta^*(u^*)} ds - Z_s^{2,u^*,\beta^*(u^*)} dB_s - H_s^{2,u^*,\beta^*(u^*)} d\tilde{N}_s, \\ Y_{\tau^\delta}^{2,u,v} = 0, \quad s \in [t, \tau^\delta], \end{cases}$$

and

$$\begin{cases} -dY_s^\delta = \left(\inf_{v \in U} \sup_{u \in U} F(s, x, 0, 0, 0, u, v) - L|Y_s^\delta| \right) ds, \\ Y_{\tau^\delta}^\delta = 0, & s \in [t, \tau^\delta], \end{cases} \quad (6.17)$$

where we denote by L the Lipschitz constant of $F(s, x, y, 0, 0, u, v)$ with respect to y . Thanks to Lemma 2.2 and (6.15) we conclude $Y_t^{2,u^*,\beta^*(u^*)} \geq Y_t^\delta$. Then from (6.16) and (6.17) it follows that

$$2C\delta^{\frac{1}{2}} \geq \frac{1}{\delta} Y_t^\delta \rightarrow \inf_{v \in V} \sup_{u \in U} F(t, x, 0, 0, 0, u, v)$$

as $\delta \rightarrow 0$. Therefore, $\inf_{v \in V} \sup_{u \in U} F(t, x, 0, 0, 0, u, v) \leq 0$. From the definition of F it now follows that $U = (U_1, U_2)$ is a viscosity supersolution of the system (4.1). We conclude the proof. \square

Let us give the proof of the Lemmas 6.14, 6.15 and 6.16, which is an adaption of those in [21] or [9] to our framework of games of the type "NAD strategy against NAD strategy". We begin with the

Proof of Lemma 6.14 We notice that

$${}^i G_{s \wedge \tau^\delta, \tau^\delta}^{t,x;u,v} [\varphi(\tau^\delta, X_{\tau^\delta}^{t,x;u,v}) 1_{N_{\tau^\delta}^{t,i}=i} + U_{m(i+1)}(\tau^\delta, X_{\tau^\delta}^{t,x;u,v}) 1_{N_{\tau^\delta}^{t,i}=m(i+1)}]$$

is defined by the following BSDE:

$$\begin{cases} -d {}^i \hat{Y}_s^{t,x;u,v} = f_i(s, X_s^{t,x;u,v}, {}^i \hat{Y}_s^{t,x;u,v}, {}^i \hat{H}_s^{t,x;u,v}, {}^i \hat{Z}_s^{t,x;u,v}, u_s, v_s) ds \\ \quad - \lambda {}^i \hat{H}_s^{t,x;u,v} ds - {}^i \hat{Z}_s^{t,x;u,v} dB_s - {}^i \hat{H}_s^{t,x;u,v} d\tilde{N}_s, & s \in [t, \tau^\delta], \\ {}^i \hat{Y}_{\tau^\delta}^{t,x;u,v} = \varphi(\tau^\delta, X_{\tau^\delta}^{t,x;u,v}) 1_{N_{\tau^\delta}^{t,i}=i} + U_{m(i+1)}(\tau^\delta, X_{\tau^\delta}^{t,x;u,v}) 1_{N_{\tau^\delta}^{t,i}=m(i+1)}, \end{cases} \quad (6.18)$$

through the following relation:

$${}^i G_{s \wedge \tau^\delta, \tau^\delta}^{t,x;u,v} [\varphi(\tau^\delta, X_{\tau^\delta}^{t,x;u,v}) 1_{N_{\tau^\delta}^{t,i}=i} + U_{m(i+1)}(\tau^\delta, X_{\tau^\delta}^{t,x;u,v}) 1_{N_{\tau^\delta}^{t,i}=m(i+1)}] = {}^i \hat{Y}_s^{t,x;u,v}, \quad s \in [t, \tau^\delta].$$

On the other hand, by applying Itô's formula to \bar{Y}_s (see the definition of \bar{Y} in Lemma 6.14), we obtain

$$\begin{aligned} d\bar{Y}_s &= \frac{\partial \varphi}{\partial s}(s, X_s^{t,x;u,v}) ds + (\nabla_x \varphi b)(s, X_s^{t,x;u,v}, u_s, v_s) ds + (\nabla_x \varphi \sigma)(s, X_s^{t,x;u,v}, u_s, v_s) dB_s \\ &\quad + \frac{1}{2} \text{tr}(\partial_{xx} \varphi \sigma \sigma^*)(s, X_s^{t,x;u,v}, u_s, v_s) ds + (U_{m(i+1)}(s, X_s^{t,x;u,v}) - \varphi(s, X_s^{t,x;u,v})) dN_s. \end{aligned} \quad (6.19)$$

From the definition of τ^δ we have

$$\begin{aligned} \bar{Y}_{\tau^\delta} &= \varphi(\tau^\delta, X_{\tau^\delta}^{t,x;u,v}) + \int_t^{\tau^\delta} (U_{m(i+1)}(r, X_r^{t,x;u,v}) - \varphi(r, X_r^{t,x;u,v})) dN_r \\ &= \varphi(\tau^\delta, X_{\tau^\delta}^{t,x;u,v}) + (U_{m(i+1)}(\tau^\delta, X_{\tau^\delta}^{t,x;u,v}) - \varphi(\tau^\delta, X_{\tau^\delta}^{t,x;u,v})) \Delta N_{\tau^\delta} \\ &= \varphi(\tau^\delta, X_{\tau^\delta}^{t,x;u,v}) + (U_{m(i+1)}(\tau^\delta, X_{\tau^\delta}^{t,x;u,v}) - \varphi(\tau^\delta, X_{\tau^\delta}^{t,x;u,v})) 1_{N_{\tau^\delta}^{t,i}=m(i+1)} \\ &= \varphi(\tau^\delta, X_{\tau^\delta}^{t,x;u,v}) 1_{N_{\tau^\delta}^{t,i}=i} + U_{m(i+1)}(\tau^\delta, X_{\tau^\delta}^{t,x;u,v}) 1_{N_{\tau^\delta}^{t,i}=m(i+1)}, \end{aligned}$$

and thus $\bar{Y}_{\tau^\delta} = {}^i \hat{Y}_{\tau^\delta}^{t,x;u,v}$. Therefore, from the above equality, (6.18) and (6.19), and the uniqueness of the solution of BSDE (6.9), it follows that

$$Y_{s \wedge \tau^\delta}^{1,u,v} = {}^i G_{s \wedge \tau^\delta, \tau^\delta}^{t,x;u,v} [\varphi(\tau^\delta, X_{\tau^\delta}^{t,x;u,v}) 1_{N_{\tau^\delta}^{t,i}=i} + U_{m(i+1)}(\tau^\delta, X_{\tau^\delta}^{t,x;u,v}) 1_{N_{\tau^\delta}^{t,i}=m(i+1)}] - \bar{Y}_{s \wedge \tau^\delta},$$

and this allows to conclude the proof. \square

Proof of Lemma 6.15 From (3.2) it follows that, for all $p \geq 2$, there exists some positive constant $C = C_p$ such that

$$\mathbb{E}[\sup_{t \leq s \leq t+\delta} |X_s^{t,x;u,v} - x|^p | \mathcal{F}_t] \leq C\delta(1 + |x|^p), \quad \mathbb{P} - a.s.,$$

uniformly in $u \in \mathcal{U}_{t,t+\delta}, v \in \mathcal{V}_{t,t+\delta}$. Let

$$\varphi(s) = F(s, x, Y_s^{2,u,v}, H_s^{2,u,v}, Z_s^{2,u,v}, u_s, v_s) - F(s, X_s^{t,x;u,v}, Y_s^{2,u,v}, H_s^{2,u,v}, Z_s^{2,u,v}, u_s, v_s).$$

Then, we have

$$|\varphi(s)| \leq C(1 + |x|^2)(|X_s^{t,x;u,v} - x| + |X_s^{t,x;u,v} - x|^3),$$

for $s \in [t, t + \delta], (t, x) \in [0, T) \times \mathbb{R}^n, u \in \mathcal{U}_{t,t+\delta}, v \in \mathcal{V}_{t,t+\delta}$. From Lemma 2.3 it follows that

$$\begin{aligned} & \mathbb{E}[\int_t^{t+\delta} (|Y_s^{1,u,v} - Y_s^{2,u,v}|^2 + |Z_s^{1,u,v} - Z_s^{2,u,v}|^2) ds | \mathcal{F}_t] + \mathbb{E}[\int_t^{t+\delta} |H_s^{1,u,v} - H_s^{2,u,v}|^2 ds | \mathcal{F}_t] \\ & \leq C\mathbb{E}[\int_t^{t+\delta} \rho^2(|X_s^{t,x;u,v} - x|) ds | \mathcal{F}_t] \\ & \leq C\delta\mathbb{E}[\sup_{t \leq s \leq t+\delta} \rho^2(|X_s^{t,x;u,v} - x|) | \mathcal{F}_t] \leq C\delta^2, \end{aligned}$$

where $\rho(r) = (1 + |x|^2)(r + r^3)$, $r \geq 0$. Therefore, we have

$$\begin{aligned} & |Y_t^{1,u,v} - Y_t^{2,u,v}| = |\mathbb{E}[(Y_t^{1,u,v} - Y_t^{2,u,v}) | \mathcal{F}_t]| \\ & = |\mathbb{E}[\int_t^{t+\delta} (F(s, X_s^{t,x;u,v}, Y_s^{1,u,v}, H_s^{1,u,v}, Z_s^{1,u,v}, u_s, v_s) \\ & \quad - F(s, x, Y_s^{2,u,v}, H_s^{2,u,v}, Z_s^{2,u,v}, u_s, v_s)) ds | \mathcal{F}_t]| \\ & \leq C\mathbb{E}[\int_t^{t+\delta} (\rho(|X_s^{t,x;u,v} - x|) + |Y_s^{1,u,v} - Y_s^{2,u,v}| + |Z_s^{1,u,v} - Z_s^{2,u,v}|) ds | \mathcal{F}_t] \\ & \quad + C\mathbb{E}[\int_t^{t+\delta} |H_s^{1,u,v} - H_s^{2,u,v}| ds | \mathcal{F}_t] \\ & \leq C\mathbb{E}[\int_t^{t+\delta} \rho(|X_s^{t,x;u,v} - x|) ds | \mathcal{F}_t] + C\delta^{\frac{1}{2}} \{ \mathbb{E}[\int_t^{t+\delta} |Y_s^{1,u,v} - Y_s^{2,u,v}|^2 ds | \mathcal{F}_t]^{\frac{1}{2}} \\ & \quad + \mathbb{E}[\int_t^{t+\delta} |Z_s^{1,u,v} - Z_s^{2,u,v}|^2 ds | \mathcal{F}_t]^{\frac{1}{2}} + \mathbb{E}[\int_t^{t+\delta} |H_s^{1,u,v} - H_s^{2,u,v}|^2 ds | \mathcal{F}_t]^{\frac{1}{2}} \} \\ & \leq C\delta^{\frac{3}{2}}. \end{aligned}$$

The desired result then follows. \square

Proof of Lemma 6.16: Since $F(s, x, \cdot, \cdot, \cdot, u, v)$ has a linear growth in (y, h, z) , uniformly in (s, x, u, v) , there exists a positive constant C independent of δ, u and v , such that, for $s \in [t, t + \delta]$,

$$|Y_s^{2,u,v}|^2 \leq C\delta, \quad \mathbb{E}[\int_s^{t+\delta} |Z_r^{2,u,v}|^2 dr | \mathcal{F}_s] \leq C\delta, \quad \mathbb{E}[\int_s^{t+\delta} |H_r^{2,u,v}|^2 dr | \mathcal{F}_s] \leq C\delta.$$

By virtue of equation (6.10) we have, for $s \in [t, t + \delta]$,

$$|Y_s^{2,u,v}| \leq \mathbb{E}[\int_s^{t+\delta} |F(r, x, Y_r^{2,u,v}, H_r^{2,u,v}, Z_r^{2,u,v}, u_r, v_r)| dr | \mathcal{F}_s]$$

$$\begin{aligned}
&\leq C\mathbb{E}\left[\int_s^{t+\delta} (1 + |x|^2 + |Y_r^{2,u,v}| + |H_r^{2,u,v}| + |Z_r^{2,u,v}|)dr|\mathcal{F}_s\right] \\
&\leq C\delta + C\sqrt{\delta}(\mathbb{E}[\int_s^{t+\delta} |Z_r^{2,u,v}|^2 dr|\mathcal{F}_s])^{\frac{1}{2}} + C\sqrt{\delta}(\mathbb{E}[\int_s^{t+\delta} |H_r^{2,u,v}|^2 dr|\mathcal{F}_s])^{\frac{1}{2}} \\
&\leq C\delta, \text{ } \mathbb{P}\text{-a.s.}
\end{aligned}$$

By applying Itô formula to $|Y_s^{2,u,v}|^2$ we conclude

$$\mathbb{E}[\int_t^{t+\delta} |Z_s^{2,u,v}|^2 ds|\mathcal{F}_t] + \mathbb{E}[\int_t^{t+\delta} |H_s^{2,u,v}|^2 ds|\mathcal{F}_t] \leq C\delta^2, \text{ } \mathbb{P}\text{-a.s.}$$

Therefore,

$$\begin{aligned}
&\mathbb{E}[\int_t^{t+\delta} |Y_s^{2,u,v}|^2 ds|\mathcal{F}_t] + \mathbb{E}[\int_t^{t+\delta} |Z_s^{2,u,v}|^2 ds|\mathcal{F}_t] + \mathbb{E}[\int_t^{t+\delta} |H_s^{2,u,v}|^2 ds|\mathcal{F}_t] \\
&\leq C\delta^2 + \delta^{\frac{1}{2}}\{\mathbb{E}[\int_t^{t+\delta} |Z_s^{2,u,v}|^2 ds|\mathcal{F}_t]\}^{\frac{1}{2}} + C\delta^{\frac{1}{2}}\{\mathbb{E}[\int_t^{t+\delta} |H_s^{2,u,v}|^2 ds|\mathcal{F}_t]\}^{\frac{1}{2}} \\
&\leq C\delta^{\frac{3}{2}}, \text{ } \mathbb{P}\text{-a.s.}
\end{aligned}$$

The proof is complete. \square

Acknowledgements. This paper is part of the author's PhD thesis supervised by Prof. Rainer Buckdahn. The author thanks Prof. Rainer Buckdahn and Prof. Catherine Rainer for helpful discussions, suggestions, comments and careful reading.

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